

Blow-up problems for the heat equation with a local nonlinear Neumann boundary condition

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Abstract

This paper estimates the blow-up time for the heat equation $u_t = \Delta u$ with a local nonlinear Neumann boundary condition: The normal derivative $\partial u / \partial n = u^q$ on Γ_1 , one piece of the boundary, while on the rest part of the boundary, $\partial u / \partial n = 0$. The motivation of the study is the partial damage to the insulation on the surface of space shuttles caused by high speed flying subjects. We prove the solution blows up in finite time and estimate both upper and lower bounds of the blow-up time in terms of the area of Γ_1 . In many other work, they need the convexity of the domain Ω and only consider the problem with $\Gamma_1 = \partial\Omega$. In this paper, we remove the convexity condition and only require $\partial\Omega$ to be C^2 . In addition, we deal with the local nonlinearity, namely Γ_1 can be just part of $\partial\Omega$.

Keywords: Blow-up time; Heat equation; Local nonlinear; Neumann boundary condition

1 Introduction and Notations

In this paper, Ω is assumed to be a bounded open set in \mathbb{R}^n ($n \geq 2$) with $\partial\Omega \in C^2$, Γ_1 and Γ_2 are two disjoint open subsets of $\partial\Omega$ with $\bar{\Gamma}_1 \cup \bar{\Gamma}_2 = \partial\Omega$, $\tilde{\Gamma} \triangleq \bar{\Gamma}_1 \cap \bar{\Gamma}_2$ is C^1 when being regarded as $\partial\Gamma_1$ or $\partial\Gamma_2$. We study the heat equation with a local nonlinear Neumann boundary condition:

$$\begin{cases} u_t(x, t) = \Delta u(x, t) & \text{in } \Omega \times (0, T], \\ \frac{\partial u}{\partial n}(x, t) = u^q(x, t) & \text{on } \Gamma_1 \times (0, T], \\ \frac{\partial u}{\partial n}(x, t) = 0 & \text{on } \Gamma_2 \times (0, T], \\ u(x, 0) = u_0(x) & \text{in } \Omega, \end{cases} \quad (1.1)$$

where $q > 1$, $u_0 \in C^1(\bar{\Omega})$, $u_0(x) \geq 0$ and $u_0(x) \not\equiv 0$. The normal derivative on the boundary is defined as following: for any $x \in \partial\Omega$, $0 < t \leq T$,

$$\frac{\partial u}{\partial n}(x, t) \triangleq \lim_{h \rightarrow 0^+} Du(x_h, t) \cdot \vec{n}(x) \text{ as long as this limit exists,} \quad (1.2)$$

where $\vec{n}(x)$ denotes the exterior unit normal vector at x and $x_h \triangleq x - h\vec{n}(x)$ for $x \in \partial\Omega$. $\partial\Omega$ being C^2 ensures that x_h belongs to Ω when h is positive and sufficiently small.

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Our work is partially motivated by the Space Shuttle Columbia disaster in 2003. When the space shuttle was launched, a piece of foam broke off from its external tank and struck the left wing causing the insulation there damaged. As a result, the shuttle disintegrated during its reentry to the atmosphere due to the enormous heat generated near the damaged part. Based on this, we start to establish the math model.

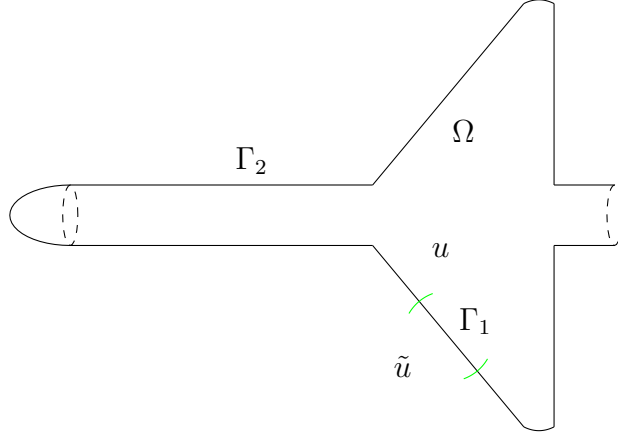


Figure 1: Model

In Figure 1, \tilde{u} represents the outside temperature of the space shuttle and u denotes the inside temperature. When the space shuttle reentered the atmosphere, it compressed the air at a very high speed. During this process, it caused many chemical reactions which produced enormous radiative heat flux. This was the main source of the heat transfer through the broken part on the left wing. In Physics, the radiation heat flux is proportional to the fourth power of the difference between the temperatures. In addition, to simplify the model, we assume $\tilde{u} = F(u)$ is an increasing function of u and treat it as a polynomial, say u^m for some $m > 1$. Thus on the broken part Γ_1 , we have

$$\frac{\partial u}{\partial n} \sim (\tilde{u} - u)^4 = (F(u) - u)^4 \sim (u^m - u)^4 \sim u^q,$$

for $q = 4m > 1$. On Γ_2 , one has $\frac{\partial u}{\partial n} = 0$, since the insulation there are intact. Inside the space shuttle, we assume it satisfies the heat equation. Thus, the realistic problem is modeled as (1.1).

The next thing is to make sense of the solution such that it exists and unique. For any $T > 0$, we define

$$\mathcal{A}_T = C^{2,1}(\Omega \times (0, T]) \cap C(\overline{\Omega} \times [0, T])$$

and

$$\mathcal{B}_T = \{g : (\Gamma_1 \cup \Gamma_2) \times (0, T] \rightarrow \mathbb{R} \mid g \in UC(\Gamma_1 \times (0, T]) \text{ and } g \in UC(\Gamma_2 \times (0, T])\},$$

where UC refers to uniformly continuous function spaces. From the definition of \mathcal{B}_T , we know for any $g \in \mathcal{B}_T$, when it is restricted to $\Gamma_i \times (0, T]$ ($i = 1$ or 2), it has a unique continuous extension to $\overline{\Gamma}_i \times [0, T]$ and we use $g|_{\Gamma_i \times (0, T]} \in C(\overline{\Gamma}_i \times [0, T])$ to denote this extension. Moreover, when there is no ambiguity, we just write $g_i \triangleq g|_{\Gamma_i \times (0, T]}$ for convenience. But one should notice that g may not extend to a continuous function on $\partial\Omega \times (0, T]$, since it can have a jump between Γ_1 and Γ_2 . Finally, we endow \mathcal{B}_T with $L^\infty((\Gamma_1 \cup \Gamma_2) \times (0, T])$ norm such that it becomes a Banach space.

The solution to (1.1) is understood in the following way.

Definition 1.1. For any $T > 0$, a solution to (1.1) on $\overline{\Omega} \times [0, T]$ means a function u in \mathcal{A}_T that satisfies

(1.1) pointwise and moreover, for any $(x, t) \in \tilde{\Gamma} \times (0, T]$, $\frac{\partial u}{\partial n}(x, t)$ exists and

$$\frac{\partial u}{\partial n}(x, t) = \frac{1}{2} u^q(x, t). \quad (1.3)$$

Here (1.3) is a technical requirement which ensures the uniqueness of the solution, see the proofs in Lemma B.7, Corollary B.8 and Theorem B.10. We will see in Section 2 that the solution to (1.1) always blows up in finite time, so we would like to study the maximal time the solution can exist.

Definition 1.2. We call

$$T^* \triangleq \sup\{T \geq 0 : \text{there exists a solution to (1.1) on } \overline{\Omega} \times [0, T]\}$$

to be the maximal existence time for (1.1). Moreover, a function $u^* \in C^{2,1}(\Omega \times (0, T^*)) \cap C(\overline{\Omega} \times [0, T^*))$ is called a maximal solution if it solves (1.1) on $\overline{\Omega} \times [0, T]$ for any $T \in (0, T^*)$.

In this paper, we denote $M_0 = \max_{\overline{\Omega}} u_0$ and write $|\Gamma_1|$ to represent the area of Γ_1 , that is

$$|\Gamma_1| = \int_{\Gamma_1} dS.$$

$\Phi : \mathbb{R}^n \times (0, \infty) \rightarrow \mathbb{R}$ represents the fundamental solution of the heat equation, that is

$$\Phi(x, t) \triangleq \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}, \quad \forall (x, t) \in \mathbb{R}^n \times (0, \infty). \quad (1.4)$$

We will show the local existence and uniqueness of the solution to (1.1). Moreover, both upper and lower bounds for the maximal existence time T^* will be given. The main results of this paper are as following.

Theorem 1.3. The maximal existence time T^* for (1.1) is positive and there exists a unique maximal solution $u^* \in C^{2,1}(\Omega \times (0, T^*)) \cap C(\overline{\Omega} \times [0, T^*))$ to (1.1). Moreover, $u^*(x, t) > 0$ for any $(x, t) \in \overline{\Omega} \times (0, T^*)$.

Theorem 1.4. Suppose T^* is the maximal existence time for (1.1), then $T^* < \infty$ and

$$\sup_{(x,t) \in \overline{\Omega} \times [0, T^*)} |u^*(x, t)| = \infty.$$

In addition, if $\min_{x \in \overline{\Omega}} u_0(x) > 0$, then

$$T^* \leq \frac{1}{(q-1)|\Gamma_1|} \int_{\Omega} u_0^{1-q}(x) dx. \quad (1.5)$$

Theorem 1.5. Suppose T^* is the maximal existence time for (1.1), then

$$T^* \geq C^{-\frac{2}{n+2}} \left[\ln \left(|\Gamma_1|^{-1} \right) - (n+2)(q-1) \ln M_0 - \ln(q-1) - \ln C \right]^{\frac{2}{n+2}}, \quad (1.6)$$

where C is a positive constant which only depends on n, Ω, q and remains bounded as $q \rightarrow 1$. As a result, no matter $|\Gamma_1| \rightarrow 0$, $M_0 \rightarrow 0$ or $q \rightarrow 1$, we will have $T^* \rightarrow \infty$.

Many work have been devoted to study the parabolic equation with Neumann boundary conditions which

are analogous to (1.1) but with $\Gamma_1 = \partial\Omega$. More precisely, they study the problem

$$\begin{cases} u_t(x, t) - \Delta u(x, t) = f(x, t) & \text{in } \Omega \times (0, T], \\ \frac{\partial u}{\partial n}(x, t) = F(x, t, u(x, t)) & \text{on } \partial\Omega \times (0, T], \\ u(x, 0) = \psi(x) & \text{in } \Omega, \end{cases} \quad (1.7)$$

where $f \in C^{\alpha, \alpha/2}(\overline{\Omega} \times [0, T])$, $F \in C(\partial\Omega \times [0, \infty) \times (-\infty, \infty))$ and $\psi \in C^1(\overline{\Omega})$. For example, [1, 2, 7, 13] discussed the existence and uniqueness of the solution to (1.7) by various methods and in different spaces. [5, 9, 11, 13, 18, 19] studied the finite time blow-up of the solution and the upper bound of the blow-up time. [14, 15, 16] estimated the lower bound of the blow-up time. [6, 7, 9, 13, 18] covered some other topics such as the localization of the blow-up points, the blow-up rate, the asymptotic behaviour near the blow-up points and so on. [4, 8, 10, 17] are books or surveys which summarized the work and methods about different issues on the problem (1.7).

However, there have not been many works on the problem (1.1) since the normal derivative $\frac{\partial u}{\partial n}$ in (1.1) is not continuous along the boundary. However, some ideas from previous work can be borrowed to apply to (1.1).

For the theories on existence and uniqueness of the solution to (1.1), one of the key tools is the Jump Relation of the Single-layer Potentials mentioned in ([7], Sec. 2, Chap. 5). We discuss several variants of this Jump Relation in Appendix A such that they can be adapted to our problem. Then we follow the arguments in [7, 13] to prove in Appendix B the existence and uniqueness of the solution to the Linear problem (B.1) and to the Nonlinear problem (B.26) by using the theories developed in Appendix A. Both proofs in Appendix A and Appendix B are very tedious and analogous to previous work, so we decide to put them into the Appendices.

To estimate the upper bound of the blow-up time, there have been existing several methods. For our problem (1.1), the simplest one to apply seems to be [18], in which it introduces a suitable energy function and shows the finite blow-up of this energy function. The process is very succinct and even gives an explicit formula for the upper bound. We follow this idea in [18] but utilize a sequence of approximated solutions $\{v_j\}_{j \geq 1}$, which satisfy the approximated problem (2.3), to justify all the calculations.

The lower bound of the blow-up time is usually harder to obtain, a popular method dealing with the lower bound is established in [15, 16]. After that, the similar idea is also applied to some more generalized problems, see e.g. [3, 12, 14]. This method also introduces a suitable energy function and derives a differential inequality for that energy function, from there the lower bound can be achieved. However, when deriving the differential inequality, their technique requires the convexity of Ω and is not applicable to the partial boundary problem, e.g. (1.1) with $\Gamma_2 \neq \emptyset$. In addition, their arguments only consider $n = 2$ or 3 . Thus, in order to handle (1.1) without any of these limitations, we seek a different way by directly analyzing the Representation formula (3.29) of u^* and taking advantage of the properties of the heat kernel. In this way, we are able to give a lower bound of the blow-up time as in Theorem 1.5.

The organization of this paper is as following: Section 2 is devoted to show Theorem 1.4. Then we prove Theorem 1.5 in Subsection 3.1 by analyzing the Representation formula (3.29) which is derived in Subsection 3.3. Subsection 3.2 compares the lower bound estimate derived from our method with previous results. Section 4 presents some numerical simulations. Appendix A introduces some generalized Jump Relations of the Single-layer Potentials. Appendix B establishes the general theories on the existence and uniqueness of the solution and verifies Theorem 1.3 as a special case.

2 Upper Bound of the Blow-up Time

First of all, we want to point out that Theorem 1.3 has been verified (See Remark B.18). Then based on this fundamental result, the goal of this section is to prove the unique solution u^* of (1.1) always blows up (i.e. L^∞ norm of u^* goes to ∞) in finite time. In addition, the blow-up time T^* is estimated in terms of $|\Gamma_1|$ and the initial data u_0 , as long as u_0 is positive on $\overline{\Omega}$.

A common way to prove the blow-up of a solution is to introduce a suitable energy function related to that solution and then derive a differential inequality to show the energy function blows up. This process usually involves integration by parts and therefore requires Du (i.e. the derivative with respect to the space variable) to be continuous up to the boundary. However, u^* is not such smooth, since the normal derivative $\frac{\partial u}{\partial n}$ is apparently not continuous along $\tilde{\Gamma}$. Thus, some approximations are needed to get through this process.

Firstly, we approximate the domain Ω from inside by $\{\Omega_k\}_{k \geq 1}$ which are defined as: for any $k \geq 1$,

$$\Omega_k \triangleq \{x \in \Omega : \text{dist}(x, \partial\Omega) > 1/k\}. \quad (2.1)$$

In addition, for any $x \in \partial\Omega_k$, we use $\vec{n}_k(x)$ to denote the exterior unit normal vector at x with respect to $\partial\Omega_k$ while for any $x \in \partial\Omega$, $\vec{n}(x)$ represents the exterior unit normal vector at x with respect to $\partial\Omega$.

Secondly, we approximate u^* by introducing a sequence of cut-off functions $\{\eta_j\}_{j \geq 1}$. More specifically, we choose a sequence of boundary pieces $\{\Gamma_{1,j}\}_{j \geq 1}$ such that $\Gamma_{1,j} \subset \Gamma_1$ and $\Gamma_{1,j} \nearrow \Gamma_1$, (see Figure 2). Then we define a sequence of C^∞ cut-off functions $\{\eta_j\}_{j=1}^\infty$ such that for each $j \geq 1$,

$$\eta_j(x) \begin{cases} = 1, & x \in \Gamma_{1,j}, \\ \in [0, 1], & x \in \overline{\Gamma_1} \setminus \Gamma_{1,j}, \\ = 0, & x \in \partial\Omega \setminus \Gamma_1. \end{cases} \quad (2.2)$$

In addition, we require that $\eta_{j+1}(x) \geq \eta_j(x)$, for any $j \geq 1$ and for any $x \in \partial\Omega$.

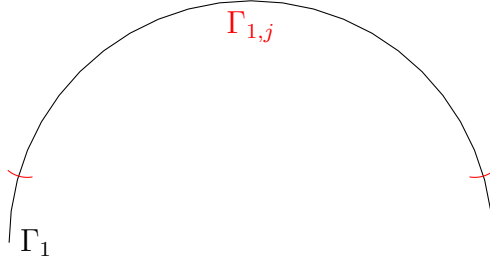


Figure 2: $\Gamma_{1,j}$

Now for each $j \geq 1$, one considers the following problem

$$\begin{cases} (v_j)_t(x, t) = \Delta v_j(x, t) & \text{in } \Omega \times (0, T], \\ \frac{\partial v_j}{\partial n}(x, t) = \eta_j(x) v_j^q(x, t) & \text{on } \partial\Omega \times (0, T], \\ v_j(x, 0) = u_0(x) & \text{in } \Omega. \end{cases} \quad (2.3)$$

If we take $f \equiv 0$, $\eta \equiv \eta_j$, $F(\lambda) = \lambda^q$ for $\lambda \in \mathbb{R}$, $\psi \equiv u_0$ and $\Gamma_2 = \emptyset$ in (B.26), then by Appendix B.2, we know (2.3) has a nonnegative unique maximal solution $v_j \in C^{2,1}(\Omega \times (0, T_j^*)) \cap C(\overline{\Omega} \times [0, T_j^*))$, where T_j^* denotes the maximal existence time for (2.3). In addition, by Corollary B.12 and (2.2), $v_j \leq u^*$ on $\overline{\Omega} \times [0, T]$ for any $T < \min\{T^*, T_j^*\}$. Thus, using Theorem B.17, one has $T_j^* \geq T^*$ and $v_j \leq u^*$ on $\overline{\Omega} \times [0, T^*)$.

Lemma 2.1. *Given $0 < t_1 < t_2 < \infty$ and suppose that $\phi : \overline{\Omega} \times [t_1, t_2] \rightarrow \mathbb{R}^n$ is continuous, then*

$$\lim_{k \rightarrow \infty} \int_{t_1}^{t_2} \int_{\partial\Omega_k} \phi(x, t) \cdot \vec{n}_k(x) dS_x dt = \int_{t_1}^{t_2} \int_{\partial\Omega} \phi(x, t) \cdot \vec{n}(x) dS_x dt.$$

Proof. Since $\partial\Omega \in C^2$, we know it has the interior ball property. Therefore when k is large enough, the mapping $\Psi_k : \partial\Omega \rightarrow \partial\Omega_k$, which is defined as

$$\Psi_k(\xi) = \xi - \frac{1}{k} \vec{n}(\xi), \quad \forall \xi \in \partial\Omega,$$

is a bijection. Moreover, one can see that Ψ_k is C^1 and $\vec{n}_k \circ \Psi_k$ is continuous on $\partial\Omega$. As a result,

$$\begin{aligned} & \int_{\partial\Omega_k} \phi(x, t) \cdot \vec{n}_k(x) dS_x \\ &= \frac{\xi = \Psi_k^{-1}(x)}{x = \Psi_k(\xi)} \int_{\partial\Omega} \phi(\Psi_k(\xi), t) \cdot \vec{n}_k(\Psi_k(\xi)) |J\Psi_k(\xi)| dS_\xi. \end{aligned}$$

Then by Lebesgue's dominated convergence theorem,

$$\lim_{k \rightarrow \infty} \int_{\partial\Omega_k} \phi(x, t) \cdot \vec{n}_k(x) dS_x = \int_{\partial\Omega} \phi(\xi, t) \cdot \vec{n}(\xi) dS_\xi.$$

In addition, we know that $\int_{\partial\Omega_k} \phi(x, t) \cdot \vec{n}_k(x) dS_x$ is uniformly bounded in k and t , since ϕ is bounded. It then again follows from Lebesgue's dominated convergence theorem that

$$\lim_{k \rightarrow \infty} \int_{t_1}^{t_2} \int_{\partial\Omega_k} \phi(x, t) \cdot \vec{n}_k(x) dS_x dt = \int_{t_1}^{t_2} \int_{\partial\Omega} \phi(\xi, t) \cdot \vec{n}(\xi) dS_\xi dt.$$

□

Now we start to prove Theorem 1.4.

Proof. Firstly by Theorem 1.3, $u^*(x, t) > 0$ for any $x \in \overline{\Omega}$ and $t > 0$, so to judge whether the solution blows up or not, we can assume

$$\inf_{x \in \Omega} u_0(x) \geq \varepsilon_0$$

for some positive constant ε_0 . Otherwise, we start from any positive time.

Secondly, for any $T \in (0, T^*)$, we fix it temporarily and denote

$$M_T \triangleq \|u^*\|_{L^\infty(\overline{\Omega} \times [0, T])}.$$

For any $j \geq 1$, recalling that v_j is the solution to (2.3), then by the maximum principle and the fact that $v_j \leq u^*$, we have

$$\varepsilon_0 \leq v_j(x, t) \leq M_T, \quad \forall (x, t) \in \overline{\Omega} \times [0, T]. \quad (2.4)$$

Moreover, it follows from ([7], the last Corollary, Sec. 4, Chap. 5) that for any $\tau_0 > 0$ and $1 \leq i \leq n$,

$$(v_j)_{x_i} \in C(\overline{\Omega} \times [\tau_0, T]). \quad (2.5)$$

Borrowing an idea from [18], for any $j \geq 1$ and $k \geq 1$, we define $h_{j,k} : [0, T] \rightarrow \mathbb{R}$ and $h_j : [0, T] \rightarrow \mathbb{R}$ by

$$h_{j,k}(t) = \int_{\Omega_k} v_j^{1-q}(x, t) dx$$

and

$$h_j(t) = \int_{\Omega} v_j^{1-q}(x, t) dx.$$

Since $v_j \in C^{2,1}(\Omega \times [0, T^*))$, the following calculations are justified.

$$\begin{aligned} h'_{j,k}(t) &= (1-q) \int_{\Omega_k} v_j^{-q}(v_j)_t dx \\ &= (1-q) \int_{\Omega_k} v_j^{-q} \Delta v_j dx \\ &= (1-q) \int_{\Omega_k} \nabla \cdot (v_j^{-q} Dv_j) + q v_j^{-q-1} |Dv_j|^2 dx \\ &\leq (1-q) \int_{\Omega_k} \nabla \cdot (v_j^{-q} Dv_j) dx \\ &= (1-q) \int_{\partial\Omega_k} v_j^{-q} Dv_j \cdot \vec{n}_k dS. \end{aligned}$$

Integrating t from τ_0 to T ,

$$h_{j,k}(T) - h_{j,k}(\tau_0) \leq (1-q) \int_{\tau_0}^T \int_{\partial\Omega_k} v_j^{-q}(x, t) Dv_j(x, t) \cdot \vec{n}_k(x) dS_x dt.$$

Taking $k \rightarrow \infty$, by (2.5) and Lemma 2.1 with $\phi = v_j^{-q} Dv_j$, we attain

$$\begin{aligned} h_j(T) - h_j(\tau_0) &\leq (1-q) \int_{\tau_0}^T \int_{\partial\Omega} v_j^{-q}(x, t) Dv_j(x, t) \cdot \vec{n}(x) dS_x dt \\ &= (1-q) \int_{\tau_0}^T \int_{\partial\Omega} \eta_j(x) dS_x dt \\ &\leq (1-q) \int_{\tau_0}^T |\Gamma_{1,j}| dt \\ &= (1-q)(T - \tau_0) |\Gamma_{1,j}|. \end{aligned}$$

Sending $\tau_0 \rightarrow 0$ and noticing (2.4), we obtain $h_j(T) - h_j(0) \leq (1-q) T |\Gamma_{1,j}|$, i.e.

$$\int_{\Omega} v_j^{1-q}(x, T) dx \leq \int_{\Omega} u_0^{1-q}(x) dx + (1-q) T |\Gamma_{1,j}|.$$

Consequently,

$$0 \leq \int_{\Omega} u_0^{1-q}(x) dx + (1-q) T |\Gamma_{1,j}|.$$

Then due to the fact that $|\Gamma_{1,j}| \rightarrow |\Gamma_1|$,

$$T \leq \frac{1}{(q-1)|\Gamma_1|} \int_{\Omega} u_0^{1-q}(x) dx.$$

Finally, since T is arbitrary in $(0, T^*)$, then

$$T^* \leq \frac{1}{(q-1)|\Gamma_1|} \int_{\Omega} u_0^{1-q}(x) dx.$$

Thus, we have shown the solution must blow up in finite time and derived an upper bound for T^* . Now it follows from Theorem B.17 that

$$\sup_{(x,t) \in \overline{\Omega} \times [0, T^*)} |u^*(x, t)| = \infty.$$

Consequently, Theorem 1.4 is verified. \square

3 Lower Bound of the Blow-up Time

3.1 Derivation of the lower bound

In this subsection, we will derive a lower bound for T^* by analyzing the Representation formulas (3.29). Here we want to point out that (3.29) is a formula for u^* on $\partial\Omega \times [0, T^*)$ which is derived from (3.28), the formula for u^* on $\Omega \times [0, T^*)$. To estimate T^* , due to the maximum principle, it suffices to study the boundary values, thus we just analyze (3.29).

The proof of (3.29) requires some work, but the idea is not hard. Firstly, if the solution u^* is smooth up to the boundary, then using integration by parts, one can easily see that u^* is also a weak solution as in Definition 3.4. From there, (3.29) can be obtained by some standard steps given in Theorem 3.8 and its Corollary 3.9. However, the solution u^* is not C^1 up to the boundary, so again we need to take an approximation procedure, which is similar to that in Section 2. This process is tedious, so we postpone it to Subsection 3.3.

Before the proof of Theorem 1.5, let's state the following Lemma 3.1 which will be used for several times in this subsection.

Lemma 3.1. *Suppose Ω is an open, bounded set in \mathbb{R}^n with $\partial\Omega \in C^2$, then there exists a constant $C = C(n, \Omega)$ such that for any $x \in \partial\Omega$ and $t > 0$,*

$$\frac{1}{t^{(n-1)/2}} \int_{\partial\Omega} e^{-\frac{|x-y|^2}{4t}} dS_y \leq C.$$

Proof. It is readily to show this conclusion by taking advantage of the definition of a C^2 boundary, we omit the proof here. \square

Now we start to prove Theorem 1.5.

Proof. In the following, C will be used to denote a positive constant which only depends on n, Ω, q and is bounded when $q \rightarrow 1$. Moreover, C may be different from line to line. We prove by analyzing the following Representation formula (3.29) for u^* on the boundary points $(x, t) \in \partial\Omega \times [0, T^*)$:

$$\begin{aligned} u^*(x, t) &= 2 \int_{\Omega} \Phi(x-y, t) u_0(y) dy + 2 \int_0^t \int_{\Gamma_1} \Phi(x-y, t-\tau) (u^*)^q(y, \tau) dS_y d\tau \\ &\quad + 2 \int_0^t \int_{\partial\Omega} (D\Phi)(x-y, t-\tau) \cdot \vec{n}(y) u^*(y, \tau) dS_y d\tau \\ &\triangleq I + II + III. \end{aligned} \tag{3.1}$$

Define $M, \widetilde{M} : [0, T^*) \rightarrow \mathbb{R}$ by

$$M(t) = \max_{y \in \partial\Omega} u^*(y, t)$$

and

$$\widetilde{M}(t) = \max_{\tau \in [0, t]} M(\tau).$$

It is clear that \widetilde{M} blows up at the same time T^* as u^* . It is also easy to see

$$I \leq 2M_0, \quad (3.2)$$

and

$$\begin{aligned} II &\leq C \int_0^t M^q(\tau) \int_{\Gamma_1} (t - \tau)^{-n/2} e^{-\frac{|x-y|^2}{4(t-\tau)}} dS_y d\tau \\ &= C \int_0^t M^q(t - \tau) \int_{\Gamma_1} \tau^{-n/2} e^{-\frac{|x-y|^2}{4\tau}} dS_y d\tau. \end{aligned} \quad (3.3)$$

In addition, we have that

$$\begin{aligned} III &\leq C \int_0^t M(\tau) \int_{\partial\Omega} \frac{|x-y|^2}{(t-\tau)^{n/2+1}} e^{-\frac{|x-y|^2}{4(t-\tau)}} dS_y d\tau \\ &= C \int_0^t M(t - \tau) \int_{\partial\Omega} \frac{|x-y|^2}{\tau^{n/2+1}} e^{-\frac{|x-y|^2}{4\tau}} dS_y d\tau \\ &\leq C \int_0^t M(t - \tau) \int_{\partial\Omega} \frac{1}{\tau^{n/2}} e^{-\frac{|x-y|^2}{8\tau}} dS_y d\tau \\ &\leq C \int_0^t M(t - \tau) \tau^{-1/2} d\tau, \end{aligned} \quad (3.4)$$

where the first inequality is due to Lemma B.2, the last inequality is due to Lemma 3.1 and the second inequality is because

$$\frac{|x-y|^2}{\tau} e^{-\frac{|x-y|^2}{8\tau}} \leq \sup_{r \geq 0} r e^{-r/8} \leq C.$$

Now we are trying to estimate (3.3) and (3.4) further. Taking $m = 1 + \frac{1}{n+1}$, it follows from Holder's inequality that

$$\begin{aligned} \int_{\Gamma_1} \tau^{-n/2} e^{-\frac{|x-y|^2}{4\tau}} dS_y &\leq \tau^{-n/2} \left(\int_{\Gamma_1} e^{-\frac{m|x-y|^2}{4\tau}} dS_y \right)^{1/m} |\Gamma_1|^{(m-1)/m} \\ &= \tau^{-\frac{n}{2} + \frac{n-1}{2m}} \left(\int_{\Gamma_1} \tau^{-\frac{n-1}{2}} e^{-\frac{m|x-y|^2}{4\tau}} dS_y \right)^{1/m} |\Gamma_1|^{(m-1)/m} \\ &\leq C \tau^{-\frac{n}{2} + \frac{n-1}{2m}} |\Gamma_1|^{\frac{m-1}{m}} \\ &= C \tau^{-\frac{2n+1}{2n+4}} |\Gamma_1|^{\frac{1}{n+2}}, \end{aligned}$$

where the second inequality is because of Lemma 3.1. Thus (3.3) leads to

$$II \leq C |\Gamma_1|^{\frac{1}{n+2}} \int_0^t M^q(t - \tau) \tau^{-\frac{2n+1}{2n+4}} d\tau. \quad (3.5)$$

By Holder's inequality again,

$$\begin{aligned}
\int_0^t M^q(t-\tau) \tau^{-\frac{2n+1}{2n+4}} d\tau &\leq \left(\int_0^t M^{\frac{qm}{m-1}}(t-\tau) d\tau \right)^{\frac{m-1}{m}} \left(\int_0^t \tau^{-\frac{(2n+1)m}{2n+4}} d\tau \right)^{\frac{1}{m}} \\
&= \left(\int_0^t M^{q(n+2)}(\tau) d\tau \right)^{\frac{1}{n+2}} \left(\int_0^t \tau^{-\frac{2n+1}{2n+2}} d\tau \right)^{\frac{n+1}{n+2}} \\
&= C t^{\frac{1}{2n+4}} \left(\int_0^t M^{q(n+2)}(\tau) d\tau \right)^{\frac{1}{n+2}}.
\end{aligned}$$

Based on this, (3.5) becomes

$$II \leq C |\Gamma_1|^{\frac{1}{n+2}} t^{\frac{1}{2n+4}} \left(\int_0^t M^{q(n+2)}(\tau) d\tau \right)^{\frac{1}{n+2}}. \quad (3.6)$$

To estimate III , it follows from (3.4) and Holder's inequality that

$$\begin{aligned}
III &\leq C \left(\int_0^t M^{n+2}(\tau) d\tau \right)^{\frac{1}{n+2}} \left(\int_0^t \tau^{-\frac{1}{2} \frac{n+2}{n+1}} d\tau \right)^{\frac{n+1}{n+2}} \\
&= C t^{\frac{n}{2n+4}} \left(\int_0^t M^{n+2}(\tau) d\tau \right)^{\frac{1}{n+2}}.
\end{aligned} \quad (3.7)$$

Combining (3.1), (3.2), (3.6), (3.7), we obtain

$$u^*(x, t) \leq C \left[M_0 + |\Gamma_1|^{\frac{1}{n+2}} t^{\frac{1}{2n+4}} \left(\int_0^t M^{q(n+2)}(\tau) d\tau \right)^{\frac{1}{n+2}} + t^{\frac{n}{2n+4}} \left(\int_0^t M^{n+2}(\tau) d\tau \right)^{\frac{1}{n+2}} \right].$$

Since x is arbitrary on $\partial\Omega$, by raising both sides to the power $n+2$,

$$M^{n+2}(t) \leq C \left[M_0^{n+2} + |\Gamma_1| t^{1/2} \int_0^t M^{q(n+2)}(\tau) d\tau + t^{n/2} \int_0^t M^{n+2}(\tau) d\tau \right].$$

As a consequence,

$$\begin{aligned}
\widetilde{M}^{n+2}(t) &\leq C \left[M_0^{n+2} + |\Gamma_1| t^{1/2} \int_0^t M^{q(n+2)}(\tau) d\tau + t^{n/2} \int_0^t M^{n+2}(\tau) d\tau \right] \\
&\leq C \left[M_0^{n+2} + |\Gamma_1| t^{1/2} \int_0^t \widetilde{M}^{q(n+2)}(\tau) d\tau + t^{n/2} \int_0^t \widetilde{M}^{n+2}(\tau) d\tau \right] \\
&\leq C(1 + t^{n/2}) \left[M_0^{n+2} + |\Gamma_1| \int_0^t \widetilde{M}^{q(n+2)}(\tau) d\tau + \int_0^t \widetilde{M}^{n+2}(\tau) d\tau \right].
\end{aligned} \quad (3.8)$$

We define

$$E(t) = M_0^{n+2} + |\Gamma_1| \int_0^t \widetilde{M}^{q(n+2)}(\tau) d\tau + \int_0^t \widetilde{M}^{n+2}(\tau) d\tau, \quad (3.9)$$

then $E(0) = M_0^{n+2}$ and $E(t)$ is increasing. Moreover $E(t)$ also blows up at T^* , since \widetilde{M} is increasing. Now (3.8) becomes

$$\widetilde{M}^{n+2}(t) \leq C(1 + t^{n/2})E(t)$$

and consequently

$$\begin{aligned} E'(t) &= |\Gamma_1| \widetilde{M}^{q(n+2)}(t) + \widetilde{M}^{(n+2)}(t) \\ &\leq C |\Gamma_1| (1 + t^{n/2})^q E^q(t) + C(1 + t^{n/2}) E(t). \end{aligned} \quad (3.10)$$

This looks like the Bernoulli equation, so we multiply both sides by $E^{-q}(t)$ and define $\Psi(t) \triangleq E^{1-q}(t)$, then $\Psi(t) \rightarrow 0$ as t approaches to T^* and

$$\Psi'(t) + C(q-1)(1 + t^{n/2})\Psi(t) \geq -C(q-1)|\Gamma_1| (1 + t^{n/2})^q. \quad (3.11)$$

We introduce the integration factor $\mu(t)$ which is defined as

$$\mu(t) \triangleq \exp \left[C \int_0^t (q-1)(1 + \tau^{n/2}) d\tau \right], \quad \forall t \geq 0.$$

It is easy to see that

$$\begin{aligned} \mu(t) &\leq \exp \left[C \left(t + t^{1+\frac{n}{2}} \right) \right] \\ &\leq \exp \left[C \left(1 + t^{1+\frac{n}{2}} \right) \right] \\ &\leq C \exp \left(C t^{1+\frac{n}{2}} \right). \end{aligned} \quad (3.12)$$

Multiplying (3.11) by $\mu(t)$, one gets

$$(\mu(t)\Psi(t))' \geq -C(q-1)|\Gamma_1| (1 + t^{n/2})^q \mu(t).$$

Integrating this inequality and noticing that $\mu(0)\Psi(0) = M_0^{-(n+2)(q-1)}$, one obtains

$$\mu(t)\Psi(t) \geq M_0^{-(n+2)(q-1)} - C(q-1)|\Gamma_1| \int_0^t (1 + \tau^{n/2})^q \mu(\tau) d\tau. \quad (3.13)$$

It follows from (3.12) that

$$\begin{aligned} \int_0^t (1 + \tau^{n/2})^q \mu(\tau) d\tau &\leq C \int_0^t (1 + \tau^{n/2})^q \exp(C\tau^{1+\frac{n}{2}}) d\tau \\ &\leq C(1 + t^{n/2})^q t \exp(Ct^{1+\frac{n}{2}}) \\ &\leq C \exp[(C+1)t^{1+\frac{n}{2}}]. \end{aligned}$$

Plugging in (3.13), we obtain

$$\mu(t)\Psi(t) \geq M_0^{-(n+2)(q-1)} - C(q-1)|\Gamma_1| \exp(Ct^{1+\frac{n}{2}}).$$

Taking $t \rightarrow T^*$, one obtains

$$\begin{aligned} C(q-1)|\Gamma_1| \exp \left[C(T^*)^{1+\frac{n}{2}} \right] &\geq M_0^{-(n+2)(q-1)} \\ \exp \left[C(T^*)^{1+\frac{n}{2}} \right] &\geq C^{-1}(q-1)^{-1}|\Gamma_1|^{-1}M_0^{-(n+2)(q-1)} \\ C(T^*)^{\frac{n+2}{2}} &\geq \ln \left(|\Gamma_1|^{-1} \right) - (n+2)(q-1) \ln M_0 - \ln(q-1) - \ln C. \end{aligned}$$

Hence,

$$T^* \geq C^{-\frac{2}{n+2}} \left[\ln \left(|\Gamma_1|^{-1} \right) - (n+2)(q-1) \ln M_0 - \ln(q-1) - \ln C \right]^{\frac{2}{n+2}}.$$

□

Remark 3.2. From Theorem 1.5, one can see that T^* goes to infinity if $|\Gamma_1| \rightarrow 0$, $M_0 \rightarrow 0$ or $q \rightarrow 1$. In addition, when $|\Gamma_1| \rightarrow 0$, the lower bound of T^* given in Theorem 1.5 is about the size

$$\left[\ln \left(|\Gamma_1|^{-1} \right) \right]^{2/(n+2)}.$$

3.2 Comparison to previous work

Although many papers study the heat equation under Dirichlet or Neumann boundary conditions, few of them deal with the lower bound of the blow-up time. One of the influential work in this area is [15], however, their method has some limitations:

- (i) $n = 2$ or 3 ;
- (ii) Ω needs to be convex;
- (iii) Only applicable to the case when $\Gamma_1 = \partial\Omega$.

As we have seen in Subsection 3.1, our method of analyzing the Representation formula does not have any of these limitations. To show another advantage of this method, we compare with [15] under the same limitations (i), (ii) and (iii).

But this time we will not directly use the result (1.6) since that estimate focuses on the behavior of T^* when $|\Gamma_1| \rightarrow 0$. As a result, although still applying Representation formula (3.29), we focus on $M_0 \triangleq \max_{\overline{\Omega}} u_0$ instead of $|\Gamma_1|$ to derive the lower bound (3.18). Then it is shown in the end of this subsection that if M_0 is big and u_0 does not change much on $\overline{\Omega}$, then our estimate (3.18) is larger than the result in [15].

In [15], when $n = 3$, the authors consider the energy function

$$\phi(t) \triangleq \int_{\Omega} (u^*)^{2m}(x, t) dx,$$

which they prove to blow up at the same time T^* as u^* for $m \geq 2q - 2$. They derive a first order differential inequality for $\phi(t)$ by using a technique developed in [16] and show $\phi(t)$ remains bounded before some time T_0 , therefore T_0 is a lower bound for T^* . From their paper, they attain the following estimate:

$$T^* \geq C \left(\int_{\Omega} u_0^{2m}(x, t) dx \right)^{-2}. \quad (3.14)$$

When $n = 2$, the arguments are similar and they obtain

$$T^* \geq C \left(\int_{\Omega} u_0^{4m}(x, t) dx \right)^{-1}. \quad (3.15)$$

Now assuming (i)-(iii) and $M_0 \geq 1$, we will estimate similarly as we did in Subsection 3.1 to obtain a lower bound, but focusing on M_0 instead of $|\Gamma_1|$, since Γ_1 has been fixed to be $\partial\Omega$ now. The following notations are as same as those in Subsection 3.1. Based on (3.9), (3.10) and noticing the assumption $|\Gamma_1| = |\partial\Omega|$, we have

$$E(0) = M_0^{n+2}$$

and

$$E'(t) \leq C \left[(1 + t^{n/2})^q E^q(t) + (1 + t^{n/2}) E(t) \right]. \quad (3.16)$$

Due to the assumption that $M_0 \geq 1$, then $E(t) \geq 1$ for all $t \geq 0$. As a consequence, (3.16) implies

$$E'(t) \leq C (1 + t^{n/2})^q E^q(t). \quad (3.17)$$

Integrating (3.17), we get

$$\begin{aligned} \int_0^t \frac{E'(\tau)}{E^q(\tau)} d\tau &\leq C \int_0^t (1 + \tau^{n/2})^q d\tau, \\ E^{1-q}(t) - M_0^{(1-q)(n+2)} &\geq (1-q) C \left(t + t^{1+\frac{nq}{2}} \right), \\ E^{1-q}(t) &\geq M_0^{-(q-1)(n+2)} - (q-1) C \left(t + t^{1+\frac{nq}{2}} \right). \end{aligned}$$

Taking $t \rightarrow T^*$,

$$(q-1) C \left[T^* + (T^*)^{1+\frac{nq}{2}} \right] \geq M_0^{-(q-1)(n+2)}.$$

If $T^* \leq 1$, then $T^* \geq (T^*)^{1+\frac{nq}{2}}$ and therefore

$$(q-1) C T^* \geq M_0^{-(q-1)(n+2)}.$$

Hence we obtain the following statement.

Theorem 3.3. *Suppose T^* is the maximal existence time for (1.1) with $\Gamma_1 = \partial\Omega$ and $M_0 \geq 1$, then*

$$T^* \geq \min \left\{ 1, M_0^{-(q-1)(n+2)} / C \right\}, \quad (3.18)$$

where C is a positive constant which only depends on n , Ω and q .

Now let's compare (3.14) and (3.15) with (3.18). If M_0 is very large and the initial function u_0 does not oscillate too much, then both (3.14) and (3.15) are about the size M_0^{-4m} . On the other hand, (3.18) is about the size $M_0^{-(q-1)(n+2)}$. Recalling that $m \geq 2q - 2$, so we have

$$4m \geq 8(q-1) \geq (q-1)(n+2),$$

for no matter $n = 2$ or 3 . Thus, our estimate for the lower bound is larger .

3.3 Weak Solution and Representation Formula

By Theorem 1.3, there exists a nonnegative unique maximal solution $u^* \in C^{2,1}(\Omega \times (0, T^*)) \cap C(\overline{\Omega} \times [0, T^*))$ to (1.1) such that

$$\begin{cases} u_t^*(x, t) = \Delta u^*(x, t) & \text{in } \Omega \times (0, T^*), \\ \frac{\partial u^*}{\partial n}(x, t) = (u^*)^q(x, t) & \text{on } \Gamma_1 \times (0, T^*), \\ \frac{\partial u^*}{\partial n}(x, t) = 0 & \text{on } \Gamma_2 \times (0, T^*), \\ u^*(x, 0) = u_0(x) & \text{in } \Omega, \end{cases} \quad (3.19)$$

where T^* is the maximal existence time as in Definition 1.2. In this subsection, we will first verify that the solution u^* to (1.1) is also a weak solution (See Definition 3.4) and then derive Representation formulas (3.28) and (3.29) for u^* .

Definition 3.4. Suppose T^* is the maximal existence time for (1.1), then a function $u \in C(\overline{\Omega} \times [0, T^*))$ is called a weak solution of (1.1) if for any $t \in (0, T^*)$ and for any $\phi \in C^2(\overline{\Omega} \times [0, t])$,

$$\begin{aligned} \int_0^t \int_{\Omega} (\phi_\tau + \Delta \phi)(y, \tau) u^*(y, \tau) dy d\tau &= \int_{\Omega} \phi(y, t) u^*(y, t) - \phi(y, 0) u_0(y) dy \\ &- \int_0^t \int_{\Gamma_1} \phi(y, \tau) (u^*)^q(y, \tau) dS_y d\tau + \int_0^t \int_{\partial\Omega} u^*(y, \tau) \frac{\partial \phi}{\partial n}(y, \tau) dS_y d\tau. \end{aligned} \quad (3.20)$$

In order to prove u^* satisfies (3.20), we are again seeking some smoother approximations for u^* . Let $\{\eta_j\}_{j \geq 1}$ be the same sequence of cut-off functions as defined in (2.2) and consider the following problem: for any $j \geq 1$,

$$\begin{cases} (w_j)_t(x, t) = \Delta w_j(x, t) & \text{in } \Omega \times (0, T), \\ \frac{\partial w_j}{\partial n}(x, t) = \eta_j(x) (u^*)^q(x, t) & \text{on } \partial\Omega \times (0, T), \\ w_j(x, 0) = u_0(x) & \text{in } \Omega. \end{cases} \quad (3.21)$$

Here we want to point out that (3.21) is different from (2.3), since (3.21) is linear in w_j while (2.3) is nonlinear in v_j . As a result, the solution of (3.21) is easier to compare with u^* .

Since u^* is only defined in $\overline{\Omega} \times [0, T^*)$, we should consider (3.21) only for $T \leq T^*$. In addition, by Subsection B.1, the maximal solution w_j to (3.21) also exists until T^* , since (3.21) is linear in w_j . Thus, we conclude that $w_j \in C^{2,1}(\Omega \times (0, T^*)) \cap C(\overline{\Omega} \times [0, T^*))$. Moreover, it follows from ([7], the last Corollary, Sec. 4, Chap. 5) that for any $1 \leq i \leq n$ and $\tau_0 > 0$,

$$\frac{\partial w_j}{\partial x_i} \in C(\overline{\Omega} \times [\tau_0, T^*)). \quad (3.22)$$

Hence, w_j has better regularity than u^* and this will help us to justify the calculations in the proof of Theorem 3.7. But before that, let's demonstrate two other basic properties of $\{w_j\}_{j \geq 1}$, namely Lemma 3.5 and Lemma 3.6.

Lemma 3.5. For any $j \geq 1$, $w_j \leq u^*$ on $\overline{\Omega} \times [0, T^*)$.

Proof. Denoting $v_j = u^* - w_j$, then for any $T \in (0, T^*)$,

$$\begin{cases} (v_j)_t(x, t) - \Delta v_j(x, t) = 0 & \text{in } \Omega \times (0, T], \\ \frac{\partial v_j}{\partial n}(x, t) = [1 - \eta_j(x)] (u^*)^q(x, t) \geq 0 & \text{on } \partial\Omega \times (0, T], \\ v_j(x, 0) = 0 & \text{in } \Omega. \end{cases}$$

By Corollary B.8, $v_j \geq 0$ on $\overline{\Omega} \times [0, T]$. □

The following pointwise convergence property is essential to justify the approximation process in the proof of Theorem 3.7.

Lemma 3.6. *For any $(x, t) \in \overline{\Omega} \times [0, T^*)$, $\lim_{j \rightarrow \infty} w_j(x, t) = u^*(x, t)$.*

Proof. Since $w_j(x, 0) = u_0(x) = u^*(x, 0)$, it suffices to prove for any $T \in (0, T^*)$,

$$\lim_{j \rightarrow \infty} w_j(x, t) = u^*(x, t), \quad \forall (x, t) \in \overline{\Omega} \times (0, T].$$

We now fix $T \in (0, T^*)$ and define $\eta^* : \Gamma_1 \cup \Gamma_2 \rightarrow \mathbb{R}$ by

$$\eta^*(x) = \begin{cases} 1, & x \in \Gamma_1, \\ 0, & x \in \Gamma_2. \end{cases}$$

Let $v_j = u^* - w_j$, then

$$\begin{cases} (v_j)_t(x, t) = \Delta v_j(x, t) & \text{in } \Omega \times (0, T], \\ \frac{\partial v_j}{\partial n}(x, t) = [\eta^*(x) - \eta_j(x)] (u^*)^q(x, t) & \text{on } (\Gamma_1 \cup \Gamma_2) \times (0, T], \\ v_j(x, 0) = 0 & \text{in } \Omega. \end{cases} \quad (3.23)$$

Similar to the proof of Theorem B.6, v_j can be written in the following form

$$v_j(x, t) = \int_0^t \int_{\partial\Omega} \Phi(x - y, t - \tau) \varphi_j(y, \tau) dS_y d\tau, \quad \forall (x, t) \in \overline{\Omega} \times (0, T], \quad (3.24)$$

where $\varphi_j \in \mathcal{B}_T$ satisfies for any $(x, t) \in (\Gamma_1 \cup \Gamma_2) \times (0, T]$,

$$\varphi_j(x, t) = \int_0^t \int_{\partial\Omega} K(x, t; y, \tau) \varphi_j(y, \tau) dS_y d\tau + H_j(x, t) \quad (3.25)$$

with

$$K(x, t; y, \tau) = -2(D\Phi)(x - y, t - \tau) \cdot \vec{n}(x)$$

and

$$H_j(x, t) = 2[\eta^*(x) - \eta_j(x)] (u^*)^q(x, t).$$

Since the function K also satisfies (B.4), we can follow the same way as the derivations of (B.18), (B.19) to obtain

$$\varphi_j(x, t) = \int_0^t \int_{\partial\Omega} K^*(x, t; y, \tau) H_j(y, \tau) dS_y d\tau + H_j(x, t) \quad (3.26)$$

for some function K^* . Moreover, there exists a constant C^* , only depending on n , Ω and T , such that

$$|K^*(x, t; y, \tau)| \leq C^*(t - \tau)^{-3/4} |x - y|^{-(n-3/2)}. \quad (3.27)$$

Due to the choice of $\{\eta_j\}_{j \geq 1}$ and the fact that u^* is bounded on $\overline{\Omega} \times [0, T]$, we know H_j is also bounded on $\overline{\Omega} \times [0, T]$ and

$$\lim_{j \rightarrow \infty} H_j(x, t) = 0, \quad \forall (x, t) \in (\Gamma_1 \cup \Gamma_2) \times (0, T].$$

Then it follows from (3.26), (3.27) and the Lebesgue's dominated convergence theorem that

$$\lim_{j \rightarrow \infty} \varphi_j(x, t) = 0, \quad \forall (x, t) \in (\Gamma_1 \cup \Gamma_2) \times (0, T].$$

In addition, the boundedness of H_j implies the boundedness of φ_j , hence by (3.24) and the Lebesgue's dominated convergence theorem again, we get

$$\lim_{j \rightarrow \infty} v_j(x, t) = 0, \quad \forall (x, t) \in \overline{\Omega} \times (0, T].$$

That is

$$\lim_{j \rightarrow \infty} w_j(x, t) = u^*(x, t), \quad \forall (x, t) \in \overline{\Omega} \times (0, T].$$

□

Next, we will verify u^* to be a weak solution by taking advantage of $\{w_j\}_{j \geq 1}$.

Theorem 3.7. *The maximal solution u^* to (1.1) is also a weak solution as in Definition 3.4.*

Proof. Firstly, we choose the same sequence of domains $\{\Omega_k\}_{k \geq 1}$ as defined in (2.1), i.e.

$$\Omega_k = \{x \in \Omega : \text{dist}(x, \partial\Omega) > 1/k\}.$$

Then for any $0 < \tau_0 < t < T^*$, $k \geq 1$, $j \geq 1$ and $\phi \in C^2(\overline{\Omega} \times [0, t])$, we have

$$\int_{\tau_0}^t \int_{\Omega_k} (w_j)_t(y, \tau) \phi(y, \tau) dy d\tau = \int_{\tau_0}^t \int_{\Omega_k} \Delta w_j(y, \tau) \phi(y, \tau) dy d\tau.$$

Using integration by parts,

$$\begin{aligned} \int_{\tau_0}^t \int_{\Omega_k} (\phi_\tau + \Delta\phi)(y, \tau) w_j(y, \tau) dy d\tau &= \int_{\Omega_k} \phi(y, t) w_j(y, t) - \phi(y, \tau_0) w_j(y, \tau_0) dy \\ &\quad - \int_{\tau_0}^t \int_{\partial\Omega_k} \phi(y, \tau) Dw_j(y, \tau) \cdot \vec{n}_k(y) dS_y d\tau + \int_{\tau_0}^t \int_{\partial\Omega_k} w_j(y, \tau) D\phi(y, \tau) \cdot \vec{n}_k(y) dS_y d\tau, \end{aligned}$$

where \vec{n}_k denotes the exterior unit normal vector with respect to $\partial\Omega_k$.

Sending $k \rightarrow \infty$, by (3.22) and Lemma 2.1, we obtain

$$\begin{aligned} \int_{\tau_0}^t \int_{\Omega} (\phi_\tau + \Delta\phi)(y, \tau) w_j(y, \tau) dy d\tau &= \int_{\Omega} \phi(y, t) w_j(y, t) - \phi(y, \tau_0) w_j(y, \tau_0) dy \\ &\quad - \int_{\tau_0}^t \int_{\partial\Omega} \phi(y, \tau) Dw_j(y, \tau) \cdot \vec{n}(y) dS_y d\tau + \int_{\tau_0}^t \int_{\partial\Omega} w_j(y, \tau) D\phi(y, \tau) \cdot \vec{n}(y) dS_y d\tau, \end{aligned}$$

Noticing $Dw_j(y, \tau) \cdot \vec{n}(y) = \frac{\partial w_j}{\partial n}(y, \tau) = \eta_j(y) (u^*)^q(y, \tau)$ on $\partial\Omega$, we get

$$\begin{aligned} \int_{\tau_0}^t \int_{\Omega} (\phi_\tau + \Delta\phi)(y, \tau) w_j(y, \tau) dy d\tau &= \int_{\Omega} \phi(y, t) w_j(y, t) - \phi(y, \tau_0) w_j(y, \tau_0) dy \\ &\quad - \int_{\tau_0}^t \int_{\partial\Omega} \phi(y, \tau) \eta_j(y) (u^*)^q(y, \tau) dS_y d\tau + \int_{\tau_0}^t \int_{\partial\Omega} w_j(y, \tau) D\phi(y, \tau) \cdot \vec{n}(y) dS_y d\tau, \end{aligned}$$

Taking $j \rightarrow \infty$, then it follows from Lemma 3.5, Lemma 3.6 and the Lebesgue's dominated convergence

theorem that

$$\begin{aligned} \int_{\tau_0}^t \int_{\Omega} (\phi_{\tau} + \Delta \phi)(y, \tau) u^*(y, \tau) dy d\tau &= \int_{\Omega} \phi(y, t) u^*(y, t) - \phi(y, \tau_0) u^*(y, \tau_0) dy \\ &- \int_{\tau_0}^t \int_{\Gamma_1} \phi(y, \tau) (u^*)^q(y, \tau) dS_y d\tau + \int_{\tau_0}^t \int_{\partial\Omega} u^*(y, \tau) D\phi(y, \tau) \cdot \vec{n}(y) dS_y d\tau. \end{aligned}$$

Finally by sending $\tau_0 \rightarrow 0$, we get (3.20). \square

Next by (3.20) and some standard steps, we are able to attain the Representation formula of u^* for inside points.

Theorem 3.8. *For the maximal solution u^* to (1.1), it has the Representation formula for the inside points $(x, t) \in \Omega \times [0, T^*)$,*

$$\begin{aligned} u^*(x, t) &= \int_{\Omega} \Phi(x - y, t) u_0(y) dy + \int_0^t \int_{\Gamma_1} \Phi(x - y, t - \tau) (u^*)^q(y, \tau) dS_y d\tau \\ &+ \int_0^t \int_{\partial\Omega} (D\Phi)(x - y, t - \tau) \cdot \vec{n}(y) u^*(y, \tau) dS_y d\tau. \end{aligned} \quad (3.28)$$

Proof. For any $x \in \Omega$, $t \in (0, T^*)$ and $\varepsilon > 0$, we define $\phi^x : \overline{\Omega} \times [0, t) \rightarrow \mathbb{R}$ by

$$\phi^x(y, \tau) = \Phi(x - y, t - \tau) = \frac{1}{(4\pi)^{n/2}} \frac{1}{(t - \tau)^{n/2}} e^{-\frac{|x-y|^2}{4(t-\tau)}}$$

and define $\phi^{x,\varepsilon} : \overline{\Omega} \times [0, t] \rightarrow \mathbb{R}$ by

$$\phi^{x,\varepsilon}(y, \tau) = \Phi(x - y, t + \varepsilon - \tau) = \frac{1}{(4\pi)^{n/2}} \frac{1}{(t + \varepsilon - \tau)^{n/2}} e^{-\frac{|x-y|^2}{4(t+\varepsilon-\tau)}}.$$

From these, one can see that $\phi^{x,\varepsilon}$ is smooth in its domain and

$$(\phi^{x,\varepsilon})_{\tau}(y, \tau) + \Delta_y(\phi^{x,\varepsilon})(y, \tau) = 0, \quad \forall (y, \tau) \in \overline{\Omega} \times [0, t].$$

Then we apply (3.20) with $\phi = \phi^{x,\varepsilon}$ to attain

$$\begin{aligned} \int_{\Omega} \phi^{x,\varepsilon}(y, t) u^*(y, t) dy &= \int_{\Omega} \phi^{x,\varepsilon}(y, 0) u_0(y) dy + \int_0^t \int_{\Gamma_1} \phi^{x,\varepsilon}(y, \tau) (u^*)^q(y, \tau) dS_y d\tau \\ &- \int_0^t \int_{\partial\Omega} \frac{\partial \phi^{x,\varepsilon}}{\partial n}(y, \tau) u^*(y, \tau) dS_y d\tau. \end{aligned}$$

Sending $\varepsilon \rightarrow 0$, it follows from the Lebesgue's dominated convergence theorem that

$$\begin{aligned} u^*(x, t) &= \int_{\Omega} \phi^x(y, 0) u_0(y) dy + \int_0^t \int_{\Gamma_1} \phi^x(y, \tau) (u^*)^q(y, \tau) dS_y d\tau \\ &- \int_0^t \int_{\partial\Omega} \frac{\partial \phi^x}{\partial n}(y, \tau) u^*(y, \tau) dS_y d\tau \\ &= \int_{\Omega} \Phi(x - y, t) u_0(y) dy + \int_0^t \int_{\Gamma_1} \Phi(x - y, t - \tau) (u^*)^q(y, \tau) dS_y d\tau \\ &+ \int_0^t \int_{\partial\Omega} (D\Phi)(x - y, t - \tau) \cdot \vec{n}(y) u^*(y, \tau) dS_y d\tau. \end{aligned}$$

The last equality is because

$$\frac{\partial \phi^x}{\partial n}(y, \tau) \triangleq D_y[\phi^x(y, \tau)] \cdot \vec{n}(y) = -(D\Phi)(x - y, t - \tau) \cdot \vec{n}(y).$$

Now we have proved (3.28) for $(x, t) \in \Omega \times (0, T^*)$, next it is obvious to see that when $t \rightarrow 0$, both sides of (3.28) tend to $u_0(x)$. Thus (3.28) is also true for $(x, 0)$, where $x \in \Omega$. \square

Theorem 3.8 only gives the formula for the inside points, but we still need the formula for the boundary points $(x, t) \in \partial\Omega \times [0, T^*)$. In order to get that, we combine Theorem 3.8 and Corollary A.2.

Corollary 3.9. *For the maximal solution u^* to (1.1), it has the Representation formula for the boundary points $(x, t) \in \partial\Omega \times [0, T^*)$,*

$$\begin{aligned} u^*(x, t) = & 2 \int_{\Omega} \Phi(x - y, t) u_0(y) dy + 2 \int_0^t \int_{\Gamma_1} \Phi(x - y, t - \tau) (u^*)^q(y, \tau) dS_y d\tau \\ & + 2 \int_0^t \int_{\partial\Omega} (D\Phi)(x - y, t - \tau) \cdot \vec{n}(y) u^*(y, \tau) dS_y d\tau. \end{aligned} \quad (3.29)$$

Proof. We fix any point $(x, t) \in \partial\Omega \times (0, T^*)$ and write $x_h = x - h\vec{n}(x)$ for $h > 0$. As shown in the proof of Corollary A.2, when h is sufficiently small, $x_h \in \Omega$ for any $x \in \partial\Omega$. Consequently we can apply Theorem 3.8 to conclude that

$$\begin{aligned} u^*(x_h, t) = & \int_{\Omega} \Phi(x_h - y, t) u_0(y) dy + \int_0^t \int_{\Gamma_1} \Phi(x_h - y, t - \tau) (u^*)^q(y, \tau) dS_y d\tau \\ & + \int_0^t \int_{\partial\Omega} (D\Phi)(x_h - y, t - \tau) \cdot \vec{n}(y) u^*(y, \tau) dS_y d\tau. \end{aligned}$$

Taking $h \rightarrow 0^+$, then it follows from Corollary A.2 that

$$\begin{aligned} u^*(x, t) = & \int_{\Omega} \Phi(x - y, t) u_0(y) dy + \int_0^t \int_{\Gamma_1} \Phi(x - y, t - \tau) (u^*)^q(y, \tau) dS_y d\tau \\ & + \int_0^t \int_{\partial\Omega} (D\Phi)(x - y, t - \tau) \cdot \vec{n}(y) u^*(y, \tau) dS_y d\tau + \frac{1}{2} u^*(x, t), \end{aligned}$$

which implies (3.29).

Now we have proved (3.29) for $(x, t) \in \partial\Omega \times (0, T^*)$. Next since we assume $\partial\Omega \in C^2$, then for any $x \in \partial\Omega$,

$$\lim_{t \rightarrow 0^+} \int_{\Omega} \Phi(x - y, t) u_0(y) dy = \frac{1}{2} u_0(x).$$

As a result, when $t \rightarrow 0$, both sides of (3.29) tend to $u_0(x)$. Thus (3.29) is also true for $(x, 0)$, where $x \in \partial\Omega$. \square

4 Numerical Simulation

We have seen from the previous sections that as $|\Gamma_1| \rightarrow 0$, the upper bound of T^* given in Theorem 1.4 is in the order $|\Gamma_1|^{-1}$ while the lower bound given in Theorem 1.5 is in the order $[\ln(|\Gamma_1|^{-1})]^{2/(n+2)}$. The natural question is which order is more accurate? In this section, we try to numerically examine the order α : $T^* \sim C |\Gamma_1|^{-\alpha}$.

However, it has some difficulties to perform the simulation, since the blow-up of the solution destroys the accuracy of the numerical schemes. As a result, to ensure the accuracy, the numerical simulation should stop at some time T_0 before u^* becomes large, say $\max_{\bar{\Omega} \times [0, T_0]} u^* = 10$. If we can show that T_0 is already in the order of $|\Gamma_1|^{-1}$ when $|\Gamma_1| \rightarrow 0$, then one can expect the order for T^* should be $|\Gamma_1|^{-1}$, since it is the same as the order of the upper bound.

In this section, we define $M_1, m_1 : [0, T^*) \rightarrow \mathbb{R}$ by

$$M_1(t) = \max_{x \in \bar{\Omega}} u^*(x, t)$$

and

$$m_1(t) = \min_{x \in \bar{\Omega}} u^*(x, t).$$

The following are the simulation results by applying the Finite Difference Method.

- 2 Dimension: Unit square, space step size $h=1/40$, time step size $k = 0.2h^2$, the length $|\Gamma_1|$ from 20/40 to 3/40;
- 3 Dimension: Unit cubic, space step size $h=1/10$, time step size $k = 0.1h^2$, the area $|\Gamma_1|$ from 49/100 to 9/100.

Two Dimensional Cases:

- Let $q = 2$ and initial data $u_0(x) \equiv 0.05$. Table 1 denotes the first time T_0 for M_1 to reach 10.

$ \Gamma_1 $	20/40	10/40	5/40	3/40
T_0	35.4	72.8	149.6	253.6
Order		1.040	1.039	1.033
$m_1(T_0)$	1.17	1.57	2.32	3.21

Table 1: 2 Dimension, $q = 2$

- Let $q = 3$ and initial data $u_0(x) \equiv 0.05$. Table 2 denotes the first time T_0 for M_1 to reach 10.

$ \Gamma_1 $	20/40	10/40	5/40	3/40
T_0	394.6	791.8	1588.5	2652.5
Order		1.005	1.005	1.004
$m_1(T_0)$	0.81	0.95	1.16	1.37

Table 2: 2 Dimension, $q = 3$

Three Dimensional Cases:

- Let $q = 2$ and initial data $u_0(x) \equiv 0.05$. Table 3 denotes the first time T_0 for M_1 to reach 10.
- Let $q = 3$ and initial data $u_0(x) \equiv 0.05$. Table 4 denotes the first time T_0 for M_1 to reach 10.

From any of these tables, the order of the blow-up time is about 1. Thus, we conjecture that the blow-up time T^* should be comparable to $|\Gamma_1|^{-1}$ as $|\Gamma_1| \rightarrow 0$.

$ \Gamma_1 $	49/100	25/100	16/100	9/100
T_0	36.3	72.6	114.7	206.9
Order		1.028	1.024	1.026
$m_1(T_0)$	1.23	1.51	1.73	2.17

Table 3: 3 Dimension, $q = 2$

$ \Gamma_1 $	49/100	25/100	16/100	9/100
T_0	403.0	791.6	1238.4	2205.4
Order		1.003	1.003	1.003
$m_1(T_0)$	0.84	0.93	1.00	1.13

Table 4: 3 Dimension, $q = 3$

Appendices

A Jump Relation

One of the key tools to show the existence of the solution to the parabolic equations with Neumann boundary conditions is the Jump Relation of the Single-layer Potentials (See [7], Sec. 2, Chap. 5). For the convenience of the readers, we restate it here.

Theorem A.1 (Jump Relation). *Let $g \in C(\partial\Omega \times [0, T])$, then for any $(x, t) \in \partial\Omega \times (0, T]$,*

$$\begin{aligned} & \lim_{h \rightarrow 0^+} \int_0^t \int_{\partial\Omega} (D\Phi)(x_h - y, t - \tau) \cdot \vec{n}(x) g(y, \tau) dS_y d\tau \\ &= \int_0^t \int_{\partial\Omega} (D\Phi)(x - y, t - \tau) \cdot \vec{n}(x) g(y, \tau) dS_y d\tau + \frac{1}{2} g(x, t), \end{aligned} \quad (\text{A.1})$$

where Φ is the fundamental solution of the heat equation, $\vec{n}(x)$ denotes the exterior unit normal vector at x with respect to $\partial\Omega$ and $x_h \triangleq x - h \vec{n}(x)$.

In this section, we discuss several variants of Theorem A.1. These variants will be mainly applied to show the existence of the solution to (B.1) and (B.26). We also have used them in some other places, for example, in the proofs of Lemma 3.6 and Corollary 3.9.

The first variant is the following Corollary A.2, in which the normal direction in the integrand is not fixed to be $\vec{n}(x)$ as in Theorem A.1.

Corollary A.2. *Let $g \in C(\partial\Omega \times [0, T])$, then for any $x \in \partial\Omega$, $t \in (0, T]$,*

$$\begin{aligned} & \lim_{h \rightarrow 0^+} \int_0^t \int_{\partial\Omega} (D\Phi)(x_h - y, t - \tau) \cdot \vec{n}(y) g(y, \tau) dS_y d\tau \\ &= \int_0^t \int_{\partial\Omega} (D\Phi)(x - y, t - \tau) \cdot \vec{n}(y) g(y, \tau) dS_y d\tau + \frac{1}{2} g(x, t). \end{aligned} \quad (\text{A.2})$$

Proof. Based on Theorem A.1, it suffices to show

$$\begin{aligned} & \lim_{h \rightarrow 0^+} \int_0^t \int_{\partial\Omega} (D\Phi)(x_h - y, t - \tau) \cdot [\vec{n}(y) - \vec{n}(x)] g(y, \tau) dS_y d\tau \\ &= \int_0^t \int_{\partial\Omega} (D\Phi)(x - y, t - \tau) \cdot [\vec{n}(y) - \vec{n}(x)] g(y, \tau) dS_y d\tau. \end{aligned} \quad (\text{A.3})$$

We denote N_x to be the normal line of $\partial\Omega$ at x . Since $\partial\Omega \in C^2$, then it satisfies the interior ball condition at x , which means that there exists an open ball $B \subset \Omega$ with center on N_x and $\bar{B} \cap \partial\Omega = \{x\}$. Hence, if we denote the radius of B to be R , then for any $h \in (0, R)$, $x_h \in N_x \cap \Omega$. In addition, $|x_h - x| \leq |x_h - y|$ for any $y \in \partial\Omega$. Thus,

$$|x - y| \leq |x - x_h| + |x_h - y| \leq 2|x_h - y|.$$

Now combining the fact $\partial\Omega \in C^2$ again, we get

$$|\vec{n}(y) - \vec{n}(x)| \leq C|y - x| \leq C|x_h - y|.$$

Consequently,

$$\begin{aligned} & \left| (D\Phi)(x_h - y, t - \tau) \cdot [\vec{n}(y) - \vec{n}(x)] g(y, \tau) \right| \\ & \leq \frac{C|x_h - y|^2}{(t - \tau)^{n/2+1}} \exp\left(-\frac{|x_h - y|^2}{4(t - \tau)}\right) \\ & \leq \frac{C}{(t - \tau)^{n/2}} \exp\left(-\frac{|x_h - y|^2}{8(t - \tau)}\right) \\ & \leq \frac{C}{(t - \tau)^{n/2}} \exp\left(-\frac{|x - y|^2}{32(t - \tau)}\right) \end{aligned}$$

This inequality enables us to apply the Lebesgue's dominated convergence theorem to verify (A.3). \square

Theorem A.1 and Corollary A.2 are still not enough for our problems. For example, in order to show the existence of the solution to (B.1), the boundary functions β and g are only assumed in \mathcal{B}_T , not in $C(\partial\Omega \times [0, T])$. Thus we need to adapt this formula to the functions in \mathcal{B}_T . The following Theorem A.3 is our observation, but the essence of the proof is the same as that of Theorem A.1.

The following are some notations needed in the proof of Theorem A.3. We write $\mathbf{0}$ and $\tilde{\mathbf{0}}$ to be the origins in \mathbb{R}^n and \mathbb{R}^{n-1} respectively and \mathbf{e}_n denotes the point $(0, 0, \dots, 0, 1)$ in \mathbb{R}^n . For any point $y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$, we write

$$\tilde{y} = (y_1, y_2, \dots, y_{n-1}).$$

For any $r > 0$,

$$B_r \triangleq B(\mathbf{0}, r)$$

means the ball in \mathbb{R}^n with radius r and

$$\tilde{B}_r \triangleq B(\tilde{\mathbf{0}}, r)$$

represents the ball in \mathbb{R}^{n-1} with radius r . Γ is used to denote the Gamma function, i.e. $\Gamma : (0, \infty) \rightarrow \mathbb{R}$ defined by

$$\Gamma(a) = \int_0^\infty t^{a-1} e^{-t} dt.$$

We should be careful to distinguish the notation Γ from the notations Γ_1 , Γ_2 and $\tilde{\Gamma}$.

Theorem A.3. *Let $\varphi \in \mathcal{B}_T$, $i = 1$ or 2 , then for any $x \in \tilde{\Gamma} \triangleq \bar{\Gamma}_1 \cap \bar{\Gamma}_2$, $t \in (0, T]$,*

$$\begin{aligned} & \lim_{h \rightarrow 0^+} \int_0^t \int_{\Gamma_i} (D\Phi)(x_h - y, t - \tau) \cdot \vec{n}(x) \varphi(y, \tau) dS_y d\tau \\ &= \int_0^t \int_{\Gamma_i} (D\Phi)(x - y, t - \tau) \cdot \vec{n}(x) \varphi(y, \tau) dS_y d\tau + \frac{1}{4} \varphi_i(x, t), \end{aligned} \quad (\text{A.4})$$

where $\varphi_i \triangleq \varphi|_{\Gamma_i \times (0, T]}$ represents the extension of φ on $\bar{\Gamma}_i \times [0, T]$.

Proof. We assume $i = 1$ (The case $i = 2$ is similar). By (1.4), (A.4) becomes

$$\begin{aligned} & \lim_{h \rightarrow 0^+} - \int_0^t \int_{\Gamma_1} \frac{(x_h - y) \cdot \vec{n}(x)}{(t - \tau)^{n/2+1}} \exp\left(-\frac{|x_h - y|^2}{4(t - \tau)}\right) \varphi_1(y, \tau) dS_y d\tau \\ &= - \int_0^t \int_{\Gamma_1} \frac{(x - y) \cdot \vec{n}(x)}{(t - \tau)^{n/2+1}} \exp\left(-\frac{|x - y|^2}{4(t - \tau)}\right) \varphi_1(y, \tau) dS_y d\tau + \frac{(4\pi)^{n/2}}{2} \varphi_1(x, t). \end{aligned} \quad (\text{A.5})$$

Without loss of generality, we assume $x = \mathbf{0}$, otherwise we can do a translation. After this, we further assume $\vec{n}(\mathbf{0}) = -\mathbf{e}_n$, otherwise we can do a rotation which preserves the dot product and the distance. By these two simplifications, we have $x = \mathbf{0}$ and $\vec{n}(x) = -\mathbf{e}_n$, therefore $x_h = h\mathbf{e}_n$ and (A.5) is reduced to

$$\begin{aligned} & \lim_{h \rightarrow 0^+} \int_0^t \int_{\Gamma_1} \frac{h - y_n}{(t - \tau)^{n/2+1}} \exp\left(-\frac{|h\mathbf{e}_n - y|^2}{4(t - \tau)}\right) \varphi_1(y, \tau) dS_y d\tau \\ &= - \int_0^t \int_{\Gamma_1} \frac{y_n}{(t - \tau)^{n/2+1}} \exp\left(-\frac{|y|^2}{4(t - \tau)}\right) \varphi_1(y, \tau) dS_y d\tau + \frac{(4\pi)^{n/2}}{2} \varphi_1(\mathbf{0}, t). \end{aligned}$$

By a change of variable in τ , it is equivalent to

$$\begin{aligned} & \lim_{h \rightarrow 0^+} \int_0^t \int_{\Gamma_1} \frac{h - y_n}{\tau^{n/2+1}} \exp\left(-\frac{|h\mathbf{e}_n - y|^2}{4\tau}\right) \varphi_1(y, t - \tau) dS_y d\tau \\ &= - \int_0^t \int_{\Gamma_1} \frac{y_n}{\tau^{n/2+1}} \exp\left(-\frac{|y|^2}{4\tau}\right) \varphi_1(y, t - \tau) dS_y d\tau + \frac{(4\pi)^{n/2}}{2} \varphi_1(\mathbf{0}, t). \end{aligned} \quad (\text{A.6})$$

Because $\partial\Omega \in C^2$ and $\tilde{\Gamma} = \partial\Gamma_1 \in C^1$, we can straighten the boundary. More specifically, after relabeling the coordinates, there exist $\phi_1 \in C^2 : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$, $\phi_2 \in C^1 : \mathbb{R}^{n-2} \rightarrow \mathbb{R}$, $\eta_0 > 0$ and a neighborhood $S_{\eta_0} \subset \partial\Omega$ of $\mathbf{0}$ such that S_{η_0} can be parametrized as

$$S_{\eta_0} = \{(\tilde{y}, \phi_1(\tilde{y})) : \tilde{y} \in \tilde{B}_{\eta_0}\}$$

and for any $y \in \tilde{\Gamma} \cap S_{\eta_0}$, we not only have $y_n = \phi_1(\tilde{y})$, but also $y_{n-1} = \phi_2(y_1, y_2, \dots, y_{n-2})$. Fixing η_0 and for any $\eta < \eta_0$, we define

$$S_\eta = \{(\tilde{y}, \phi_1(\tilde{y})) : \tilde{y} \in \tilde{B}_\eta\},$$

which is a subset of S_{η_0} and also a small neighborhood of $\mathbf{0}$. Then we denote

$$S_{\eta,1} = S_\eta \cap \Gamma_1, \quad \tilde{S}_\eta = S_\eta \cap \tilde{\Gamma},$$

$$\tilde{B}_{\eta,1} = \{\tilde{y} \in \tilde{B}_\eta : (\tilde{y}, \phi_1(\tilde{y})) \in S_{\eta,1}\}, \quad P_\eta = \{\tilde{y} \in \tilde{B}_\eta : (\tilde{y}, \phi_1(\tilde{y})) \in \tilde{S}_\eta\}.$$

After these preparations, we begin the technical proof. Given any $\varepsilon > 0$, we want to find $\delta = \delta(\varepsilon) > 0$ such

that for any $0 < h < \delta$, the difference between the two sides of (A.6) is within $C\varepsilon$ for some constant C .

For any $\eta \in (0, \eta_0)$ which will be determined later, we split the integral over Γ_1 in (A.6) into two parts: $\int_{\Gamma_1} = \int_{S_{\eta,1}} + \int_{\Gamma_1 \setminus S_{\eta,1}}$. Since $\Gamma_1 \setminus S_{\eta,1}$ is away from $\mathbf{0}$, it is easy to see there exists $\delta_1 = \delta_1(\eta, \varepsilon)$ such that when $0 < h < \delta_1$, then

$$\begin{aligned} & \left| \int_0^t \int_{\Gamma_1 \setminus S_{\eta,1}} \frac{h - y_n}{\tau^{n/2+1}} \exp\left(-\frac{|h\mathbf{e}_n - y|^2}{4\tau}\right) \varphi_1(y, t - \tau) dS_y d\tau \right. \\ & \left. + \int_0^t \int_{\Gamma_1 \setminus S_{\eta,1}} \frac{y_n}{\tau^{n/2+1}} \exp\left(-\frac{|y|^2}{4\tau}\right) \varphi_1(y, t - \tau) dS_y d\tau \right| < \varepsilon. \end{aligned} \quad (\text{A.7})$$

Next since $\vec{n}(\mathbf{0}) = -\mathbf{e}_n$, then $D\phi_1(\tilde{\mathbf{0}}) = \tilde{\mathbf{0}}$. As a result, for any $y \in S_{\eta,1}$,

$$\begin{aligned} |y_n| &= |\phi_1(\tilde{y})| = |\phi_1(\tilde{y}) - \phi_1(\tilde{\mathbf{0}})| = |D\phi_1(\theta\tilde{y}) \cdot \tilde{y}| \\ &\leq |D\phi_1(\theta\tilde{y}) - D\phi_1(\tilde{\mathbf{0}})| |\tilde{y}| \leq C |\tilde{y}|^2, \end{aligned} \quad (\text{A.8})$$

where we used the mean value theorem twice. By (A.8), together with the fact $|h\mathbf{e}_n - y| \geq |\tilde{y}|$, we attain

$$|y_n| \exp\left(-\frac{|h\mathbf{e}_n - y|^2}{4\tau}\right) \leq C |\tilde{y}|^2 \exp\left(-\frac{|\tilde{y}|^2}{4\tau}\right).$$

Noticing

$$\int_0^t \int_{S_{\eta,1}} \frac{|\tilde{y}|^2}{\tau^{n/2+1}} \exp\left(-\frac{|\tilde{y}|^2}{4\tau}\right) dS_y d\tau < \infty,$$

then it follows from Lebesgue's dominated convergence theorem that

$$\begin{aligned} & \lim_{h \rightarrow 0^+} \int_0^t \int_{S_{\eta,1}} \frac{y_n}{\tau^{n/2+1}} \exp\left(-\frac{|h\mathbf{e}_n - y|^2}{4\tau}\right) \varphi_1(y, t - \tau) dS_y d\tau \\ &= \int_0^t \int_{S_{\eta,1}} \frac{y_n}{\tau^{n/2+1}} \exp\left(-\frac{|y|^2}{4\tau}\right) \varphi_1(y, t - \tau) dS_y d\tau. \end{aligned}$$

As a result, there exists $\delta_2 = \delta_2(\eta, \varepsilon)$ such that when $0 < h < \delta_2$, then

$$\begin{aligned} & \left| \int_0^t \int_{S_{\eta,1}} \frac{y_n}{\tau^{n/2+1}} \exp\left(-\frac{|h\mathbf{e}_n - y|^2}{4\tau}\right) \varphi_1(y, t - \tau) dS_y d\tau \right. \\ & \left. - \int_0^t \int_{S_{\eta,1}} \frac{y_n}{\tau^{n/2+1}} \exp\left(-\frac{|y|^2}{4\tau}\right) \varphi_1(y, t - \tau) dS_y d\tau \right| < \varepsilon. \end{aligned} \quad (\text{A.9})$$

Now it suffices to verify that $|I_\eta(h, t) - \frac{1}{2} (4\pi)^{n/2} \varphi_1(\mathbf{0}, t)| < C\varepsilon$, where

$$I_\eta(h, t) \triangleq \int_0^t \int_{S_{\eta,1}} \frac{h}{\tau^{n/2+1}} \exp\left(-\frac{|h\mathbf{e}_n - y|^2}{4\tau}\right) \varphi_1(y, t - \tau) dS_y d\tau. \quad (\text{A.10})$$

Recalling that $y_n = \phi(\tilde{y})$, (A.10) can be rewritten as

$$\begin{aligned} I_\eta(h, t) &\triangleq \int_0^t \int_{\tilde{B}_{\eta,1}} \frac{h}{\tau^{n/2+1}} \exp\left(-\frac{|h\mathbf{e}_n - y|^2}{4\tau}\right) \varphi_1(y, t - \tau) \sqrt{1 + |D\phi_1(\tilde{y})|^2} d\tilde{y} d\tau \\ &= \int_0^t \int_{\tilde{B}_{\eta,1}} \frac{h}{\tau^{n/2+1}} \exp\left(-\frac{|\tilde{y}|^2 + |h - y_n|^2}{4\tau}\right) \varphi_1(y, t - \tau) \sqrt{1 + |D\phi_1(\tilde{y})|^2} d\tilde{y} d\tau, \end{aligned} \quad (\text{A.11})$$

where $y = (\tilde{y}, \phi_1(\tilde{y}))$. I_η is hard to compute, so we approximate it by a simpler function. We define $\tilde{I}_\eta(h, t)$ as following

$$\begin{aligned}\tilde{I}_\eta(h, t) &\triangleq \int_0^t \int_{\tilde{B}_{\eta,1}} \frac{h}{\tau^{n/2+1}} \exp\left(-\frac{|h\mathbf{e}_n - (\tilde{y}, 0)|^2}{4\tau}\right) \varphi_1(\mathbf{0}, t - \tau) d\tilde{y} d\tau \\ &= \int_0^t \int_{\tilde{B}_{\eta,1}} \frac{h}{\tau^{n/2+1}} \exp\left(-\frac{|\tilde{y}|^2 + h^2}{4\tau}\right) \varphi_1(\mathbf{0}, t - \tau) d\tilde{y} d\tau.\end{aligned}\tag{A.12}$$

Our strategy is to show that $\tilde{I}_\eta(h, t)$ is close to both $\frac{1}{2} (4\pi)^{n/2} \varphi_1(\mathbf{0}, t)$ and $I_\eta(h, t)$.

Based on (A.12), we first reverse the order of integration and then make the change of variable $\tau \rightarrow \sigma = (|\tilde{y}|^2 + h^2)/(4\tau)$ to obtain

$$\begin{aligned}\tilde{I}_\eta(h, t) &\triangleq \int_{\tilde{B}_{\eta,1}} \int_0^t \frac{h}{\tau^{n/2+1}} \exp\left(-\frac{|\tilde{y}|^2 + h^2}{4\tau}\right) \varphi_1(\mathbf{0}, t - \tau) d\tau d\tilde{y} \\ &= \int_{\tilde{B}_{\eta,1}} \int_{\frac{|\tilde{y}|^2 + h^2}{4t}}^\infty \frac{4^{n/2} \sigma^{n/2-1} h}{(|\tilde{y}|^2 + h^2)^{n/2}} e^{-\sigma} \varphi_1\left(\mathbf{0}, t - \frac{|\tilde{y}|^2 + h^2}{4\sigma}\right) d\sigma d\tilde{y} \\ &= 4^{n/2} \int_{\tilde{B}_{\eta,1}} \frac{h}{(|\tilde{y}|^2 + h^2)^{n/2}} \int_{\frac{|\tilde{y}|^2 + h^2}{4t}}^\infty \sigma^{n/2-1} e^{-\sigma} \varphi_1\left(\mathbf{0}, t - \frac{|\tilde{y}|^2 + h^2}{4\sigma}\right) d\sigma d\tilde{y} \\ &= 4^{n/2} \int_{\tilde{B}_{\eta,1}} \frac{h}{(|\tilde{y}|^2 + h^2)^{n/2}} H(|\tilde{y}|^2 + h^2, t) d\tilde{y},\end{aligned}\tag{A.13}$$

where

$$H(\lambda, t) \triangleq \int_{\frac{\lambda}{4t}}^\infty \sigma^{n/2-1} e^{-\sigma} \varphi_1\left(\mathbf{0}, t - \frac{\lambda}{4\sigma}\right) d\sigma.$$

It is readily to see that

$$\lim_{\lambda \rightarrow 0} H(\lambda, t) = \Gamma\left(\frac{n}{2}\right) \varphi_1(\mathbf{0}, t).\tag{A.14}$$

Consequently, there exists $\delta_3 = \delta_3(\varepsilon)$ such that when $\eta < \delta_3$ and $0 < h < \delta_3$, then

$$\left| H(|\tilde{y}|^2 + h^2, t) - \Gamma\left(\frac{n}{2}\right) \varphi_1(\mathbf{0}, t) \right| < \varepsilon, \quad \forall \tilde{y} \in \tilde{B}_{\eta,1}.\tag{A.15}$$

After having taken care of the H term in (A.13), let's consider the following integration

$$\int_{\tilde{B}_{\eta,1}} \frac{h}{(|\tilde{y}|^2 + h^2)^{n/2}} d\tilde{y},$$

where the integrand $h(|\tilde{y}|^2 + h^2)^{-n/2}$ is radian in $\tilde{y} \in \mathbb{R}^{n-1}$ and positive when $h > 0$. Since $\tilde{\Gamma} = \partial\Gamma_1 \in C^1$, it ensures that P_η almost bisects \tilde{B}_η when η is small, which means $\tilde{B}_{\eta,1}$ is close to a hemisphere and

$$\lim_{\eta \rightarrow 0} \frac{|\tilde{B}_{\eta,1}|}{|\tilde{B}_\eta|} = \frac{1}{2}.$$

As a result, we can find $\delta_4 = \delta_4(\varepsilon)$ such that for any $\eta < \delta_4$,

$$1 - \varepsilon < \frac{\int_{\tilde{B}_{\eta,1}} \frac{h}{(|\tilde{y}|^2 + h^2)^{n/2}} d\tilde{y}}{\frac{1}{2} \int_{\tilde{B}_\eta} \frac{h}{(|\tilde{y}|^2 + h^2)^{n/2}} d\tilde{y}} < 1 + \varepsilon,$$

i.e.

$$\left| \int_{\tilde{B}_{\eta,1}} \frac{h}{(|\tilde{y}|^2 + h^2)^{n/2}} d\tilde{y} - \frac{1}{2} \int_{\tilde{B}_\eta} \frac{h}{(|\tilde{y}|^2 + h^2)^{n/2}} d\tilde{y} \right| < \frac{\varepsilon}{2} \int_{\tilde{B}_\eta} \frac{h}{(|\tilde{y}|^2 + h^2)^{n/2}} d\tilde{y}. \quad (\text{A.16})$$

Next, we will estimate $\int_{\tilde{B}_\eta} \frac{h}{(|\tilde{y}|^2 + h^2)^{n/2}} d\tilde{y}$. Making the change of variable $\tilde{y} \rightarrow \tilde{z} \triangleq \tilde{y}/h$,

$$\int_{\tilde{B}_\eta} \frac{h}{(|\tilde{y}|^2 + h^2)^{n/2}} d\tilde{y} = \int_{\tilde{B}_{\eta/h}} \frac{1}{(|\tilde{z}|^2 + 1)^{n/2}} d\tilde{z}.$$

On one hand,

$$\int_{\tilde{B}_{\eta/h}} \frac{1}{(|\tilde{z}|^2 + 1)^{n/2}} d\tilde{z} \leq \int_{\mathbb{R}^{n-1}} \frac{1}{(|\tilde{z}|^2 + 1)^{n/2}} d\tilde{z} = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2})},$$

while on the other hand,

$$\lim_{h \rightarrow 0} \int_{\tilde{B}_{\eta/h}} \frac{1}{(|\tilde{z}|^2 + 1)^{n/2}} d\tilde{z} = \int_{\mathbb{R}^{n-1}} \frac{1}{(|\tilde{z}|^2 + 1)^{n/2}} d\tilde{z} = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2})}.$$

Thus, there exists $\delta_5 = \delta_5(\eta, \varepsilon)$ such that for any $0 < h < \delta_5$,

$$\left| \int_{\tilde{B}_\eta} \frac{h}{(|\tilde{y}|^2 + h^2)^{n/2}} d\tilde{y} - \frac{\pi^{n/2}}{\Gamma(\frac{n}{2})} \right| < \varepsilon \quad (\text{A.17})$$

and therefore by (A.16),

$$\left| \int_{\tilde{B}_{\eta,1}} \frac{h}{(|\tilde{y}|^2 + h^2)^{n/2}} d\tilde{y} - \frac{\pi^{n/2}}{2\Gamma(\frac{n}{2})} \right| < C\varepsilon. \quad (\text{A.18})$$

It then follows from (A.13), (A.15) and (A.18) that

$$\left| \tilde{I}_\eta(h, t) - \frac{(4\pi)^{n/2}}{2} \varphi_1(\mathbf{0}, t) \right| < C\varepsilon. \quad (\text{A.19})$$

Now it left to show that $\tilde{I}_\eta(h, t)$ is close to $I_\eta(h, t)$. Firstly, because of (A.8), $|\tilde{y}|^2 + |h - y_n|^2$ is comparable to $|\tilde{y}|^2 + h^2$. More precisely, there exist positive constants $m_1 < 1$ and $M_1 > 1$ such that

$$m_1 (|\tilde{y}|^2 + h^2) \leq |\tilde{y}|^2 + |h - y_n|^2 \leq M_1 (|\tilde{y}|^2 + h^2). \quad (\text{A.20})$$

We can equivalently write it to be

$$m_1 |h\mathbf{e}_n - (\tilde{y}, 0)|^2 \leq |h\mathbf{e}_n - y|^2 \leq M_1 |h\mathbf{e}_n - (\tilde{y}, 0)|^2. \quad (\text{A.21})$$

Next, it follows from (A.11) and (A.12) that

$$\begin{aligned}
& |I_\eta(h, t) - \tilde{I}_\eta(h, t)| \\
& \leq \int_0^t \frac{h}{\tau^{n/2+1}} \int_{\tilde{B}_{\eta,1}} \left| e^{-\frac{|h\mathbf{e}_n - y|^2}{4\tau}} - e^{-\frac{|h\mathbf{e}_n - (\tilde{y}, 0)|^2}{4\tau}} \right| \left| \varphi_1(y, t - \tau) \sqrt{1 + |D\phi_1(\tilde{y})|^2} \right| d\tilde{y} d\tau \\
& \quad + \int_0^t \frac{h}{\tau^{n/2+1}} \int_{\tilde{B}_{\eta,1}} e^{-\frac{|h\mathbf{e}_n - (\tilde{y}, 0)|^2}{4\tau}} \left| \varphi_1(y, t - \tau) \sqrt{1 + |D\phi_1(\tilde{y})|^2} - \varphi_1(\mathbf{0}, t - \tau) \right| d\tilde{y} d\tau \\
& \triangleq I + II,
\end{aligned} \tag{A.22}$$

where $y = (\tilde{y}, \phi_1(\tilde{y}))$. For II , since $\varphi_1 \in C(\overline{\Gamma}_1 \times [0, T])$, then there exists $\delta_6 = \delta_6(\varepsilon)$ such that when $\eta < \delta_6$,

$$\left| \varphi_1(y, t - \tau) \sqrt{1 + |D\phi_1(\tilde{y})|^2} - \varphi_1(\mathbf{0}, t - \tau) \right| < \varepsilon, \quad \forall \tilde{y} \in \tilde{B}_{\eta,1}, \tau \in [0, t].$$

As a result,

$$\begin{aligned}
II & \leq \varepsilon \int_0^t \frac{h}{\tau^{n/2+1}} \int_{\tilde{B}_{\eta,1}} e^{-\frac{|h\mathbf{e}_n - (\tilde{y}, 0)|^2}{4\tau}} d\tilde{y} d\tau \\
& = \varepsilon \int_0^t \frac{h}{\tau^{n/2+1}} \int_{\tilde{B}_{\eta,1}} e^{-\frac{|\tilde{y}|^2 + h^2}{4\tau}} d\tilde{y} d\tau \\
& = \varepsilon \int_{\tilde{B}_{\eta,1}} h \int_0^t \frac{1}{\tau^{n/2+1}} e^{-\frac{|\tilde{y}|^2 + h^2}{4\tau}} d\tau d\tilde{y} \\
& \leq \varepsilon \int_{\tilde{B}_{\eta,1}} h \int_0^\infty \frac{4^{n/2}}{(|\tilde{y}|^2 + h^2)^{n/2}} \sigma^{n/2-1} e^{-\sigma} d\sigma d\tilde{y} \\
& = C \varepsilon \int_{\tilde{B}_{\eta,1}} \frac{h}{(|\tilde{y}|^2 + h^2)^{n/2}} d\tilde{y},
\end{aligned}$$

where the second inequality is due to the change of variable $\tau \rightarrow \sigma \triangleq \frac{|\tilde{y}|^2 + h^2}{4\tau}$. Now by another change of variabel $\tilde{y} \rightarrow \tilde{z} \triangleq \tilde{y}/h$, we get

$$II \leq C \varepsilon \int_{\mathbb{R}^{n-1}} \frac{1}{(|\tilde{z}|^2 + 1)^{n/2}} d\tilde{z} = C\varepsilon. \tag{A.23}$$

To estimate I , firstly it is easy to see that for any $h > 0$ and $y \in \tilde{B}_{\eta,1}$,

$$h \leq |h\mathbf{e}_n - (\tilde{y}, 0)|. \tag{A.24}$$

Then by (A.8),

$$\begin{aligned}
& \left| |h\mathbf{e}_n - y|^2 - |h\mathbf{e}_n - (\tilde{y}, 0)|^2 \right| \\
& = |(h - y_n)^2 - h^2| \\
& = |y_n| |2h - y_n| \\
& \leq C |\tilde{y}|^2 (2h + |\tilde{y}|^2) \\
& \leq C |h\mathbf{e}_n - (\tilde{y}, 0)|^3.
\end{aligned}$$

Now it follows from the mean value theorem and (A.21) that

$$\begin{aligned} \left| e^{-\frac{|h\mathbf{e}_n - y|^2}{4\tau}} - e^{-\frac{|h\mathbf{e}_n - (\tilde{y}, 0)|^2}{4\tau}} \right| &\leq \frac{1}{4\tau} e^{-\frac{m_1 |h\mathbf{e}_n - (\tilde{y}, 0)|^2}{4\tau}} \left| |h\mathbf{e}_n - y|^2 - |h\mathbf{e}_n - (\tilde{y}, 0)|^2 \right| \\ &\leq C \frac{|h\mathbf{e}_n - (\tilde{y}, 0)|^3}{\tau} e^{-\frac{m_1 |h\mathbf{e}_n - (\tilde{y}, 0)|^2}{4\tau}} \end{aligned} \quad (\text{A.25})$$

Thus, based on (A.24) and (A.25), we attain

$$I \leq \int_0^t \int_{\tilde{B}_{\eta,1}} \frac{1}{\tau^{n/2+2}} e^{-\frac{m_1 |h\mathbf{e}_n - (\tilde{y}, 0)|^2}{4\tau}} |h\mathbf{e}_n - (\tilde{y}, 0)|^4 d\tilde{y} d\tau.$$

Again, by reversing the order of integration and the change of variable $\tau \rightarrow \sigma \triangleq \frac{|h\mathbf{e}_n - (\tilde{y}, 0)|^2}{4\tau}$, we get

$$\begin{aligned} I &\leq \int_{\tilde{B}_{\eta,1}} |h\mathbf{e}_n - (\tilde{y}, 0)|^4 \int_0^t \frac{1}{\tau^{n/2+2}} e^{-\frac{m_1 |h\mathbf{e}_n - (\tilde{y}, 0)|^2}{4\tau}} d\tau d\tilde{y} \\ &\leq C \int_{\tilde{B}_{\eta,1}} \frac{1}{|h\mathbf{e}_n - (\tilde{y}, 0)|^{n-2}} \int_0^\infty \sigma^{n/2} e^{-m_1 \sigma} d\sigma d\tilde{y} \\ &\leq C \int_{\tilde{B}_{\eta,1}} \frac{1}{|\tilde{y}|^{n-2}} d\tilde{y}. \end{aligned}$$

Hence, there exists $\delta_7 = \delta_7(\varepsilon)$ such that when $\eta < \delta_7$, then

$$I < \varepsilon. \quad (\text{A.26})$$

Combining (A.26) and (A.23), we get

$$|I_\eta(h, t) - \tilde{I}_\eta(h, t)| < C\varepsilon.$$

Therefore, we finish the proof.

In summary, for any $\varepsilon > 0$, we firstly determine $\delta_3(\varepsilon), \delta_4(\varepsilon), \delta_6(\varepsilon), \delta_7(\varepsilon)$ and choose $\eta < \min\{\eta_0, \delta_3, \delta_4, \delta_6, \delta_7\}$. Then we determine $\delta_1(\eta, \varepsilon), \delta_2(\eta, \varepsilon), \delta_5(\eta, \varepsilon)$ and choose $\delta < \min\{\delta_1, \delta_2, \delta_3, \delta_5\}$. Such δ is what we desire, because we can see from the above proof that for any $0 < h < \delta$, the difference between the two sides of (A.4) is less than $C\varepsilon$ for some constant C . \square

Corollary A.4. *Let $\varphi \in \mathcal{B}_T$, $i = 1$ or 2 , then for any $x \in \Gamma_i$, $t \in (0, T]$,*

$$\begin{aligned} &\lim_{h \rightarrow 0^+} \int_0^t \int_{\Gamma_i} (D\Phi)(x_h - y, t - \tau) \cdot \vec{n}(x) \varphi(y, \tau) dS_y d\tau \\ &= \int_0^t \int_{\Gamma_i} (D\Phi)(x - y, t - \tau) \cdot \vec{n}(x) \varphi(y, \tau) dS_y d\tau + \frac{1}{2} \varphi(x, t). \end{aligned} \quad (\text{A.27})$$

Proof. The proof is almost the same as that of Theorem A.3. The only difference is this time x is an inner point of Γ_i , as a result, the jump term becomes $\frac{1}{2} \varphi(x, t)$ instead of $\frac{1}{4} \varphi(x, t)$. \square

Corollary A.5. *Let $\varphi \in \mathcal{B}_T$, then for any $x \in \Gamma_1 \cup \Gamma_2$, $t \in (0, T]$,*

$$\begin{aligned} & \lim_{h \rightarrow 0^+} \int_0^t \int_{\partial\Omega} (D\Phi)(x_h - y, t - \tau) \cdot \vec{n}(x) \varphi(y, \tau) dS_y d\tau \\ &= \int_0^t \int_{\partial\Omega} (D\Phi)(x - y, t - \tau) \cdot \vec{n}(x) \varphi(y, \tau) dS_y d\tau + \frac{1}{2} \varphi(x, t). \end{aligned}$$

Proof. Without loss of generality, we suppose $x \in \Gamma_1$, then by Corollary A.4,

$$\begin{aligned} & \lim_{h \rightarrow 0^+} \int_0^t \int_{\Gamma_1} (D\Phi)(x_h - y, t - \tau) \cdot \vec{n}(x) \varphi(y, \tau) dS_y d\tau \\ &= \int_0^t \int_{\Gamma_1} (D\Phi)(x - y, t - \tau) \cdot \vec{n}(x) \varphi(y, \tau) dS_y d\tau + \frac{1}{2} \varphi(x, t). \end{aligned}$$

In addition, since the distance between x and Γ_2 is positive, then it is easy to see that

$$\begin{aligned} & \lim_{h \rightarrow 0^+} \int_0^t \int_{\Gamma_2} (D\Phi)(x_h - y, t - \tau) \cdot \vec{n}(x) \varphi(y, \tau) dS_y d\tau \\ &= \int_0^t \int_{\Gamma_2} (D\Phi)(x - y, t - \tau) \cdot \vec{n}(x) \varphi(y, \tau) dS_y d\tau. \end{aligned}$$

Adding these two equations together, the Corollary follows. \square

B Existence and Uniqueness

B.1 Linear Case

In this subsection, we will show the existence and uniqueness of the solution to the following linear initial-boundary value problem:

$$\begin{cases} u_t(x, t) - \Delta u(x, t) = f(x, t) & \text{in } \Omega \times (0, T], \\ \frac{\partial u}{\partial n}(x, t) + \beta(x, t)u(x, t) = g(x, t) & \text{on } (\Gamma_1 \cup \Gamma_2) \times (0, T], \\ u(x, 0) = \psi(x) & \text{in } \Omega, \end{cases} \quad (\text{B.1})$$

where $f \in C^{\alpha, \alpha/2}(\overline{\Omega} \times [0, T])$, $\beta, g \in \mathcal{B}_T$, $\psi \in C^1(\overline{\Omega})$. We will first show the existence and then use the existence to verify the uniqueness. In the following, Γ is also used to denote the Gamma function, i.e. $\Gamma : (0, \infty) \rightarrow \mathbb{R}$ defined by

$$\Gamma(a) = \int_0^\infty t^{a-1} e^{-t} dt.$$

For any $T > 0$, we write

$$D_{\Omega, T} = \{(x, t; y, \tau) \mid x, y \in \Omega, x \neq y, 0 \leq \tau < t \leq T\} \quad (\text{B.2})$$

to denote the domain of $K_j (j \geq 0)$ which will be constructed in the proof of Theorem B.6. The solution to (B.1) is understood in the following way.

Definition B.1. *For any $T > 0$, a solution to (B.1) on $\overline{\Omega} \times [0, T]$ means a function u in \mathcal{A}_T that satisfies*

(B.1) pointwise and moreover, for any $(x, t) \in \tilde{\Gamma} \times (0, T]$, $\frac{\partial u}{\partial n}(x, t)$ exists and

$$\frac{\partial u}{\partial n}(x, t) + \frac{1}{2} [\beta_1(x, t) + \beta_2(x, t)] u(x, t) = \frac{1}{2} [g_1(x, t) + g_2(x, t)], \quad (\text{B.3})$$

where β_i and g_i denote the extensions of β and g on $\bar{\Gamma}_i \times [0, T]$ for $i = 1$ or 2 .

Before showing the existence of the solution, we state some basic properties.

Lemma B.2. Suppose Ω is an open bounded set in \mathbb{R}^n with $\partial\Omega \in C^2$, then there exists a constant $C > 0$ such that for any $x, y \in \partial\Omega$,

$$|(x - y) \cdot \vec{n}(x)| \leq C |x - y|^2.$$

Proof. It is easy to show this conclusion by taking advantage of the definition of a C^2 boundary. \square

Lemma B.3. Suppose Ω is an open, bounded set in \mathbb{R}^n with $\partial\Omega \in C^2$, $0 \leq a < n - 1$, $0 \leq b < n - 1$, then there exists a constant C , only depending on a, b, n, Ω , such that for any $x, z \in \partial\Omega$,

$$\int_{\partial\Omega} \frac{dS_y}{|x - y|^a |y - z|^b} \leq \begin{cases} C |x - z|^{n-1-a-b} & \text{if } a + b > n - 1, \\ C & \text{if } a + b < n - 1. \end{cases}$$

Proof. See ([7], Lemma 1, Sec. 2, Chap. 5). \square

The following Lemma is mentioned in [7] and it is an important technique used in Theorem B.6, Theorem B.14 and Lemma 3.6.

Lemma B.4. Let $K_0 : D_{T,\Omega} \rightarrow \mathbb{R}$ and suppose there is a constant C such that for any $(x, t; y, \tau) \in D_{T,\Omega}$,

$$|K_0(x, t; y, \tau)| \leq C (t - \tau)^{-3/4} |x - y|^{-(n-3/2)}. \quad (\text{B.4})$$

For any $j \geq 1$, we define $K_j : D_{T,\Omega} \rightarrow \mathbb{R}$ by

$$K_j(x, t; y, \tau) \triangleq \int_{\tau}^t \int_{\partial\Omega} K_0(x, t; z, \sigma) K_{j-1}(z, \sigma; y, \tau) dS_z d\sigma. \quad (\text{B.5})$$

Then all the $K_j (j \geq 1)$ are well-defined and the series $\sum_{j=0}^{\infty} |K_j|$ converges uniformly to some function \tilde{K} on $D_{T,\Omega}$. Moreover, there exists some constant C^* , only depending on n, Ω and T , such that for any $(x, t; y, \tau) \in D_{T,\Omega}$,

$$\tilde{K}(x, t; y, \tau) \leq C^* (t - \tau)^{-3/4} |x - y|^{-(n-3/2)}. \quad (\text{B.6})$$

Proof. We can mimic the arguments from Page 14 to Page 15 in [7] to prove this Lemma, provided we take advantage of (B.4) and Lemma B.3. Also see the proof of Theorem 2 in ([7], Sec. 3, Chap. 5). \square

Lemma B.5. If $f \in C^{\alpha, \alpha/2}(\bar{\Omega} \times [0, T])$ and

$$W(x, t) \triangleq \int_0^t \int_{\Omega} \Phi(x - y, t - \tau) f(y, \tau) dS_y d\tau, \quad \forall (x, t) \in \bar{\Omega} \times [0, T],$$

then $W \in C^{2,1}(\Omega \times (0, T])$ and

$$(W_t - \Delta W)(x, t) = f(x, t), \quad \forall (x, t) \in \Omega \times (0, T].$$

Proof. See ([7], Theorem 9, Sec. 5, Chap. 1). □

Now based on the arguments in ([7], Theorem 2, Sec. 3, Chap. 5), we can prove the following existence theorem.

Theorem B.6. *For any $T > 0$, there exists a solution $u \in \mathcal{A}_T$ to (B.1) on $\overline{\Omega} \times [0, T]$.*

Proof. We will construct a solution u to (B.1). Firstly, since $\psi \in C^1(\overline{\Omega})$ and $\partial\Omega \in C^2$, one can extend ψ to a larger domain. More precisely, there exists an open set $\Omega_1 \supset \overline{\Omega}$ and $\psi_1 \in C^1(\overline{\Omega}_1)$ such that ψ_1 agrees with ψ on $\overline{\Omega}$. In the rest of the proof, for convenience, we just write ψ_1 to be ψ and therefore $\psi \in C^1(\overline{\Omega}_1)$. We are looking for a solution u in the following form: for any $(x, t) \in \overline{\Omega} \times [0, T]$,

$$\begin{aligned} u(x, t) \triangleq & \int_{\Omega_1} \Phi(x - y, t) \psi(y) dy + \int_0^t \int_{\Omega} \Phi(x - y, t - \tau) f(y, \tau) dy d\tau \\ & + \int_0^t \int_{\partial\Omega} \Phi(x - y, t - \tau) \varphi(y, \tau) dS_y d\tau, \end{aligned} \quad (\text{B.7})$$

where $\varphi \in \mathcal{B}_T$ will be determined later.

Due to Lemma B.5, it is readily to see that the function u defined in (B.7) belongs to \mathcal{A}_T and satisfies the first and the third equations in (B.1), so in order to verify u to be the solution, the only things left to check are

$$\frac{\partial u}{\partial n}(x, t) + \beta(x, t)u(x, t) = g(x, t), \quad \forall (x, t) \in (\Gamma_1 \cup \Gamma_2) \times (0, T] \quad (\text{B.8})$$

and

$$\frac{\partial u}{\partial n}(x, t) + \frac{1}{2} [\beta_1(x, t) + \beta_2(x, t)] u(x, t) = \frac{1}{2} [g_1(x, t) + g_2(x, t)], \quad \forall (x, t) \in \widetilde{\Gamma} \times (0, T]. \quad (\text{B.9})$$

The plan is to firstly find a function $\varphi \in \mathcal{B}_T$ such that u defined in (B.7) satisfies (B.8), then we will prove this u satisfies (B.9) as well.

By (1.2) and (B.7), for any $(x, t) \in (\Gamma_1 \cup \Gamma_2) \times (0, T]$,

$$\begin{aligned} \frac{\partial u}{\partial n}(x, t) &= \lim_{h \rightarrow 0^+} Du(x_h, t) \cdot \vec{n}(x) \\ &= \lim_{h \rightarrow 0^+} \left[\int_{\Omega_1} (D\Phi)(x_h - y, t) \cdot \vec{n}(x) \psi(y) dy \right. \\ &\quad + \int_0^t \int_{\Omega} (D\Phi)(x_h - y, t - \tau) \cdot \vec{n}(x) f(y, \tau) dy d\tau \\ &\quad \left. + \int_0^t \int_{\partial\Omega} (D\Phi)(x_h - y, t - \tau) \cdot \vec{n}(x) \varphi(y, \tau) dS_y d\tau \right]. \end{aligned}$$

Applying the Lebesgue's dominated convergence theorem and Corollary A.5, we get

$$\begin{aligned} \frac{\partial u}{\partial n}(x, t) &= \int_{\Omega_1} (D\Phi)(x - y, t) \cdot \vec{n}(x) \psi(y) dy + \int_0^t \int_{\Omega} (D\Phi)(x - y, t - \tau) \cdot \vec{n}(x) f(y, \tau) dy d\tau \\ &\quad + \int_0^t \int_{\partial\Omega} (D\Phi)(x - y, t - \tau) \cdot \vec{n}(x) \varphi(y, \tau) dS_y d\tau + \frac{1}{2} \varphi(x, t). \end{aligned} \quad (\text{B.10})$$

Therefore, (B.8) is reduced to for any $(x, t) \in (\Gamma_1 \cup \Gamma_2) \times (0, T]$,

$$\varphi(x, t) = \int_0^t \int_{\partial\Omega} K(x, t; y, \tau) \varphi(y, \tau) dS_y d\tau + H(x, t), \quad (\text{B.11})$$

where

$$K(x, t; y, \tau) = -2 \left[(D\Phi)(x - y, t - \tau) \cdot \vec{n}(x) + \beta(x, t) \Phi(x - y, t - \tau) \right]$$

and

$$\begin{aligned} H(x, t) &= -2 \int_{\Omega_1} \left[(D\Phi)(x - y, t) \cdot \vec{n}(x) + \beta(x, t) \Phi(x - y, t) \right] \psi(y) dy \\ &\quad - 2 \int_0^t \int_{\Omega} \left[(D\Phi)(x - y, t - \tau) \cdot \vec{n}(x) + \beta(x, t) \Phi(x - y, t - \tau) \right] f(y, \tau) dy d\tau \\ &\quad + 2g(x, t). \end{aligned}$$

Thus, the proof of (B.8) becomes the search for a fixed point $\varphi \in \mathcal{B}_T$ of (B.11).

In the following, we will construct a fixed point $\varphi \in \mathcal{B}_T$ of (B.11). By Lemma B.2, we get for any $x, y \in \partial\Omega$, $0 \leq \tau < t \leq T$,

$$\begin{aligned} |K(x, t; y, \tau)| &\leq C \left[\frac{1}{(t - \tau)^{n/2}} + \frac{|x - y|^2}{(t - \tau)^{n/2+1}} \right] e^{-\frac{|x - y|^2}{4(t - \tau)}} \\ &\leq C (t - \tau)^{-3/4} |x - y|^{-(n-3/2)}. \end{aligned} \quad (\text{B.12})$$

Then using the fact $\psi \in C^1(\overline{\Omega}_1)$ and the integration by parts, we obtain

$$\begin{aligned} \int_{\Omega_1} (D\Phi)(x - y, t) \cdot \vec{n}(x) \psi(y) dy &= - \int_{\Omega_1} D_y [\Phi(x - y, t)] \psi(y) dy \cdot \vec{n}(x) \\ &= - \int_{\partial\Omega_1} \Phi(x - y, t) \psi(y) \vec{n}(y) dy \cdot \vec{n}(x) + \int_{\Omega_1} \Phi(x - y, t) D\psi(y) dy \cdot \vec{n}(x). \end{aligned} \quad (\text{B.13})$$

Consequently, as a function in (x, t) ,

$$\int_{\Omega_1} (D\Phi)(x - y, t) \cdot \vec{n}(x) \psi(y) dy \in C(\partial\Omega \times [0, T]) \subset \mathcal{B}_T.$$

Then it is readily to check that $H \in \mathcal{B}_T$.

Next, we define $\varphi_0(x, t) = H(x, t)$ on $(\Gamma_1 \cup \Gamma_2) \times (0, T]$ and for any $j \geq 1$, we define $\varphi_j : (\Gamma_1 \cup \Gamma_2) \times (0, T] \rightarrow \mathbb{R}$ by

$$\varphi_j(x, t) = \int_0^t \int_{\partial\Omega} K(x, t; y, \tau) \varphi_{j-1}(y, \tau) dS_y d\tau + H(x, t). \quad (\text{B.14})$$

Because of (B.4), we can prove by induction that for any $j \geq 0$, φ_j is well defined and belongs to \mathcal{B}_T . The next goal is to show that $\varphi_j(x, t)$ uniformly converges to some function $\varphi(x, t)$ on $(\Gamma_1 \cup \Gamma_2) \times (0, T]$ as $j \rightarrow \infty$, which makes φ to be the fixed point of (B.11) in \mathcal{B}_T .

To show the uniform convergence of $\{\varphi_j\}_{j \geq 0}$, we define a sequence of functions $\{K_j\}_{j \geq 0}$ on $D_{\Omega, T}$ as following. For any $(x, t; y, \tau) \in D_{\Omega, T}$,

$$K_0(x, t; y, \tau) \triangleq K(x, t; y, \tau).$$

For any $j \geq 1$ and $(x, t; y, \tau) \in D_{\Omega, T}$,

$$K_j(x, t; y, \tau) \triangleq \int_{\tau}^t \int_{\partial\Omega} K(x, t; z, \sigma) K_{j-1}(z, \sigma; y, \tau) dS_z d\sigma. \quad (\text{B.15})$$

Based on (B.14) and (B.15), again by induction, one can prove that for any $j \geq 1$ and for any $(x, t) \in$

$(\Gamma_1 \cup \Gamma_2) \times (0, T]$,

$$\varphi_j(x, t) = \varphi_{j-1}(x, t) + \int_0^t \int_{\partial\Omega} K_{j-1}(x, t; y, \tau) H(y, \tau) dS_y d\tau,$$

which implies

$$\varphi_j(x, t) = \sum_{i=0}^{j-1} \int_0^t \int_{\partial\Omega} K_i(x, t; y, \tau) H(y, \tau) dS_y d\tau + H(x, t). \quad (\text{B.16})$$

Writing

$$K^*(x, t; y, \tau) \triangleq \sum_{j=0}^{\infty} K_j(x, t; y, \tau), \quad (\text{B.17})$$

by Lemma B.4, we know K^* is well-defined and $\sum_{j=0}^{\infty} K_j$ converges uniformly to K^* on $D_{T,\Omega}$. Moreover, there exists a constant $C^* = C^*(n, \Omega, T)$ such that for any $(x, t; y, \tau) \in D_{\Omega, T}$,

$$|K^*(x, t; y, \tau)| \leq C^*(t - \tau)^{-3/4} |x - y|^{-(n-3/2)}. \quad (\text{B.18})$$

Consequently, it follows from (B.16) and (B.17) that φ_j converges uniformly to the function φ on $(\Gamma_1 \cup \Gamma_2) \times (0, T]$, where

$$\varphi(x, t) \triangleq \int_0^t \int_{\partial\Omega} K^*(x, t; y, \tau) H(y, \tau) dS_y d\tau + H(x, t), \quad \forall (x, t) \in (\Gamma_1 \cup \Gamma_2) \times (0, T]. \quad (\text{B.19})$$

Thus, φ is a fixed point of (B.11) in \mathcal{B}_T and therefore the function u defined in (B.7) satisfies (B.8).

Now as our plan, it only left to confirm this function u satisfies (B.9) as well. Making use of (B.7), (1.2) and Theorem A.3, we get for any $x \in \bar{\Gamma}$, $0 < t \leq T$, $\frac{\partial u}{\partial n}(x, t)$ exists and

$$\begin{aligned} \frac{\partial u}{\partial n}(x, t) &= \int_{\Omega_1} (D\Phi)(x - y, t) \cdot \vec{n}(x) \psi(y) dy + \int_0^t \int_{\Omega} (D\Phi)(x - y, t - \tau) \cdot \vec{n}(x) f(y, \tau) dy d\tau \\ &\quad + \int_0^t \int_{\partial\Omega} (D\Phi)(x - y, t - \tau) \cdot \vec{n}(x) \varphi(y, \tau) dS_y d\tau + \frac{1}{4} \varphi_1(x, t) + \frac{1}{4} \varphi_2(x, t). \end{aligned} \quad (\text{B.20})$$

Then we choose two sequences of points $\{\xi_k\}_{k \geq 1} \subset \Gamma_1$ and $\{z_j\}_{j \geq 1} \subset \Gamma_2$ which converge to x , it follows from (B.10) that

$$\begin{aligned} \frac{\partial u}{\partial n}(\xi_k, t) &= \int_{\Omega_1} (D\Phi)(\xi_k - y, t) \cdot \vec{n}(\xi_k) \psi(y) dy \\ &\quad + \int_0^t \int_{\Omega} (D\Phi)(\xi_k - y, t - \tau) \cdot \vec{n}(\xi_k) f(y, \tau) dy d\tau \\ &\quad + \int_0^t \int_{\partial\Omega} (D\Phi)(\xi_k - y, t - \tau) \cdot \vec{n}(\xi_k) \varphi(y, \tau) dS_y d\tau + \frac{1}{2} \varphi(\xi_k, t) \end{aligned}$$

and

$$\begin{aligned} \frac{\partial u}{\partial n}(z_j, t) &= \int_{\Omega_1} (D\Phi)(z_j - y, t) \cdot \vec{n}(z_j) \psi(y) dy \\ &\quad + \int_0^t \int_{\Omega} (D\Phi)(z_j - y, t - \tau) \cdot \vec{n}(z_j) f(y, \tau) dy d\tau \\ &\quad + \int_0^t \int_{\partial\Omega} (D\Phi)(z_j - y, t - \tau) \cdot \vec{n}(z_j) \varphi(y, \tau) dS_y d\tau + \frac{1}{2} \varphi(z_j, t). \end{aligned}$$

Taking $k \rightarrow \infty$ and $j \rightarrow \infty$, we obtain

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{\partial u}{\partial n}(\xi_k, t) &= \int_{\Omega_1} (D\Phi)(x - y, t) \cdot \vec{n}(x) \psi(y) dy \\ &\quad + \int_0^t \int_{\Omega} (D\Phi)(x - y, t - \tau) \cdot \vec{n}(x) f(y, \tau) dy d\tau \\ &\quad + \int_0^t \int_{\partial\Omega} (D\Phi)(x - y, t - \tau) \cdot \vec{n}(x) \varphi(y, \tau) dS_y d\tau + \frac{1}{2} \varphi_1(x, t) \end{aligned} \quad (\text{B.21})$$

and

$$\begin{aligned} \lim_{j \rightarrow \infty} \frac{\partial u}{\partial n}(z_j, t) &= \int_{\Omega_1} (D\Phi)(x - y, t) \cdot \vec{n}(x) \psi(y) dy \\ &\quad + \int_0^t \int_{\Omega} (D\Phi)(x - y, t - \tau) \cdot \vec{n}(x) f(y, \tau) dy d\tau \\ &\quad + \int_0^t \int_{\partial\Omega} (D\Phi)(x - y, t - \tau) \cdot \vec{n}(x) \varphi(y, \tau) dS_y d\tau + \frac{1}{2} \varphi_2(x, t). \end{aligned} \quad (\text{B.22})$$

Adding (B.21) and (B.22) together and noticing (B.20), we attain

$$\frac{\partial u}{\partial n}(x, t) = \frac{1}{2} \left[\lim_{k \rightarrow \infty} \frac{\partial u}{\partial n}(\xi_k, t) + \lim_{j \rightarrow \infty} \frac{\partial u}{\partial n}(z_j, t) \right]. \quad (\text{B.23})$$

Moreover, since u satisfies (B.8), we have

$$\begin{aligned} \frac{\partial u}{\partial n}(\xi_k, t) + \beta(\xi_k, t) u(\xi_k, t) &= g(\xi_k, t), \\ \frac{\partial u}{\partial n}(z_j, t) + \beta(z_j, t) u(z_j, t) &= g(z_j, t). \end{aligned}$$

Sending $k \rightarrow \infty$ and $j \rightarrow \infty$, we obtain

$$\lim_{k \rightarrow \infty} \frac{\partial u}{\partial n}(\xi_k, t) = g_1(x, t) - \beta_1(x, t) u(x, t) \quad (\text{B.24})$$

$$\lim_{j \rightarrow \infty} \frac{\partial u}{\partial n}(z_j, t) = g_2(x, t) - \beta_2(x, t) u(x, t) \quad (\text{B.25})$$

Combining (B.23), (B.24) and (B.25) together, (B.9) follows. \square

Next, we will prove the comparison principle and the uniqueness of the solution by applying Theorem B.6. But before that, let's prove the following easier comparison result.

Lemma B.7. *Suppose in (B.1), $f \geq 0$ on $\overline{\Omega} \times [0, T]$, $\psi > 0$ on $\overline{\Omega}$ and*

$$\inf_{(\Gamma_1 \cup \Gamma_2) \times [0, T]} g > 0,$$

then the solution $u > 0$ on $\overline{\Omega} \times [0, T]$.

Proof. Since $\psi > 0$ on $\overline{\Omega}$, we have $m \triangleq \min_{\overline{\Omega}} \psi > 0$. Now we claim $u > 0$ on $\overline{\Omega} \times [0, T]$. If not, then there will exist $x_0 \in \overline{\Omega}$ and $t_0 \in (0, T]$ such that

$$u(x_0, t_0) = 0 = \min_{\overline{\Omega} \times [0, t_0]} u.$$

By the strong maximum principle, $x_0 \in \partial\Omega$. If $x_0 \in \Gamma_1 \cup \Gamma_2$, then

$$0 < g(x_0, t_0) = \frac{\partial u}{\partial n}(x_0, t_0) + \beta(x_0, t_0) u(x_0, t_0) = \frac{\partial u}{\partial n}(x_0, t_0) \leq 0,$$

which is impossible. If $x_0 \in \tilde{\Gamma}$, then

$$\begin{aligned} 0 &< \frac{1}{2} [g_1(x_0, t_0) + g_2(x_0, t_0)] \\ &= \frac{\partial u}{\partial n}(x_0, t_0) + \frac{1}{2} [\beta_1(x_0, t_0) + \beta_2(x_0, t_0)] u(x_0, t_0) \\ &= \frac{\partial u}{\partial n}(x_0, t_0) \leq 0, \end{aligned}$$

which is also a contradiction. Thus, the Lemma follows. \square

Corollary B.8. *Suppose in (B.1), $f \geq 0$ on $\overline{\Omega} \times [0, T]$, $\psi \geq 0$ on $\overline{\Omega}$ and $g \geq 0$ on $(\Gamma_1 \cup \Gamma_2) \times (0, T]$, then the solution $u \geq 0$ on $\overline{\Omega} \times [0, T]$. In particular, the solution to (B.1) on $\overline{\Omega} \times [0, T]$ is unique.*

Proof. Due to Theorem B.6, there exists a solution $v \in \mathcal{A}_T$ to the following problem:

$$\begin{cases} v_t(x, t) - \Delta v(x, t) = 1 & \text{in } \Omega \times (0, T], \\ \frac{\partial v}{\partial n}(x, t) + \beta(x, t)v(x, t) = 1 & \text{on } (\Gamma_1 \cup \Gamma_2) \times (0, T], \\ v(x, 0) = 1 & \text{in } \Omega. \end{cases}$$

For any $\varepsilon > 0$, we define $w_\varepsilon = u + \varepsilon v$, then w_ε satisfies

$$\begin{cases} (w_\varepsilon)_t(x, t) - \Delta w_\varepsilon(x, t) = f + \varepsilon \geq \varepsilon & \text{in } \Omega \times (0, T], \\ \frac{\partial w_\varepsilon}{\partial n}(x, t) + \beta(x, t)w_\varepsilon(x, t) = g + \varepsilon \geq \varepsilon & \text{on } (\Gamma_1 \cup \Gamma_2) \times (0, T], \\ w_\varepsilon(x, 0) = \psi + \varepsilon \geq \varepsilon & \text{in } \Omega. \end{cases}$$

By applying Lemma B.7, $w_\varepsilon \geq 0$ on $\overline{\Omega} \times [0, T]$. Taking $\varepsilon \rightarrow 0$, we get $u \geq 0$ on $\overline{\Omega} \times [0, T]$. \square

B.2 Nonlinear Case

This subsection is devoted to the existence and uniqueness of the solution to the following problem with a local nonlinear Neumann boundary condition:

$$\begin{cases} u_t(x, t) - \Delta u(x, t) = f(x, t) & \text{in } \Omega \times (0, T], \\ \frac{\partial u}{\partial n}(x, t) = \eta(x)F(u(x, t)) & \text{on } \Gamma_1 \times (0, T], \\ \frac{\partial u}{\partial n}(x, t) = 0 & \text{on } \Gamma_2 \times (0, T], \\ u(x, 0) = \psi(x) & \text{in } \Omega, \end{cases} \quad (\text{B.26})$$

where $f \in C^{\alpha, \alpha/2}(\overline{\Omega} \times [0, T])$, $\eta \in C^1(\overline{\Gamma}_1)$ and $\eta \geq 0$, $F \in C^1(\mathbb{R})$, $\psi \in C^1(\overline{\Omega})$. The solution is understood in the following way.

Definition B.9. *For any $T > 0$, a solution to (B.26) on $\overline{\Omega} \times [0, T]$ means a function u in \mathcal{A}_T that satisfies (B.26) pointwise and moreover, for any $(x, t) \in \tilde{\Gamma} \times (0, T]$, $\frac{\partial u}{\partial n}(x, t)$ exists and*

$$\frac{\partial u}{\partial n}(x, t) = \frac{1}{2} \eta(x) F(u(x, t)). \quad (\text{B.27})$$

This time, we will first show some comparison principles and then discuss the existence of the solution.

Theorem B.10. Suppose $u_i \in \mathcal{A}_T (i = 1, 2)$ is the solution to (B.26) on $\overline{\Omega} \times [0, T]$ with right hand side $f_i (i = 1, 2)$, $\eta F_i (i = 1, 2)$ and $\psi_i (i = 1, 2)$. If $f_1 \geq f_2$ on $\overline{\Omega} \times [0, T]$, $F_1 \geq F_2$ on \mathbb{R} , $\psi_1 \geq \psi_2$ on $\overline{\Omega}$ and $\eta \geq 0$ on $\overline{\Gamma}_1$, then $u_1 \geq u_2$ on $\overline{\Omega} \times [0, T]$. As a consequence, the solution to (B.26) is unique.

Proof. Let $f = f_1 - f_2$, $\psi = \psi_1 - \psi_2$ and $w = u_1 - u_2$, then we have $f \geq 0$ on $\overline{\Omega} \times [0, T]$, $\psi \geq 0$ on $\overline{\Omega}$ and $F_2(u_1(x, t)) - F_2(u_2(x, t)) = \beta(x, t)w(x, t)$, where

$$\beta(x, t) = \int_0^1 F_2'(s u_1(x, t) + (1 - s)u_2(x, t)) ds.$$

Thus w satisfies the following equations

$$\begin{cases} w_t(x, t) - \Delta w(x, t) = f(x, t) \geq 0 & \text{in } \Omega \times (0, T], \\ \frac{\partial w}{\partial n}(x, t) - \eta(x) \beta(x, t) w(x, t) = \eta(x) [F_1(u_1(x, t)) - F_2(u_1(x, t))] \geq 0 & \text{on } \Gamma_1 \times (0, T], \\ \frac{\partial w}{\partial n}(x, t) = 0 & \text{on } \Gamma_2 \times (0, T], \\ w(x, 0) = \psi(x) \geq 0 & \text{in } \Omega. \end{cases}$$

Now it follows from Corollary B.8 that $w \geq 0$. □

Theorem B.11. Suppose $u \in \mathcal{A}_T$ is the solution to (B.26) with $f \geq 0$ on $\overline{\Omega} \times [0, T]$, $\psi \geq 0$ on $\overline{\Omega}$, $\eta \geq 0$ on $\overline{\Gamma}_1$ and $F(0) \geq 0$, then $u \geq 0$ on $\overline{\Omega} \times [0, T]$. In addition, if $\psi \not\equiv 0$, then $u(x, t) > 0$, $\forall x \in \overline{\Omega}$, $0 < t \leq T$.

Proof. To prove the first statement, we write

$$F(u(x, t)) = F(u(x, t)) - F(0) + F(0) = \beta(x, t)u(x, t) + F(0),$$

where

$$\beta(x, t) = \int_0^1 F'(su(x, t)) ds.$$

Hence u satisfies

$$\begin{cases} u_t(x, t) - \Delta u(x, t) = f(x, t) \geq 0 & \text{in } \Omega \times (0, T], \\ \frac{\partial u}{\partial n}(x, t) - \eta(x) \beta(x, t) u(x, t) = \eta(x) F(0) \geq 0 & \text{on } \Gamma_1 \times (0, T], \\ \frac{\partial u}{\partial n}(x, t) = 0 & \text{on } \Gamma_2 \times (0, T], \\ u(x, 0) = \psi(x) \geq 0 & \text{in } \Omega. \end{cases}$$

It then follows from Corollary B.8 that $u \geq 0$. Now in order to prove the second statement, we suppose additionally that $\psi \not\equiv 0$, then by applying the strong maximum principle and the Hopf lemma, we get $u(x, t) > 0$, $\forall x \in \overline{\Omega}$, $0 < t \leq T$. □

Corollary B.12. Suppose $u_i \in \mathcal{A}_T (i = 1, 2)$ is the solution to (B.26) on $\overline{\Omega} \times [0, T]$ with right hand side f , $\eta_i F$ and ψ . If $f \geq 0$ on $\overline{\Omega} \times [0, T]$, $\psi \geq 0$ on $\overline{\Omega}$, $F(t) \geq 0$ for $t \geq 0$ and $\eta_1 \geq \eta_2 \geq 0$ on $\overline{\Gamma}_1$, then $u_1 \geq u_2$ on $\overline{\Omega} \times [0, T]$.

Proof. Firstly, by Theorem B.11, u_2 is nonnegative on $\overline{\Omega} \times [0, T]$ and therefore $F \circ u_2$ is nonnegative on

$\overline{\Omega} \times [0, T]$. Writing $w = u_1 - u_2$, then for any $x \in \Gamma_1$, $t \in (0, T]$,

$$\begin{aligned} & \eta_1(x)F(u_1(x, t)) - \eta_2(x)F(u_2(x, t)) \\ &= \eta_1(x)[F(u_1(x, t)) - F(u_2(x, t))] + F(u_2(x, t))[\eta_1(x) - \eta_2(x)] \\ &\geq \eta_1(x)[F(u_1(x, t)) - F(u_2(x, t))] \\ &= \eta_1(x)\beta(x, t)w(x, t), \end{aligned}$$

where

$$\beta(x, t) = \int_0^1 F'(s u_1(x, t) + (1-s)u_2(x, t)) ds.$$

Thus, w satisfies

$$\begin{cases} w_t(x, t) - \Delta w(x, t) = 0 & \text{in } \Omega \times (0, T], \\ \frac{\partial w}{\partial n}(x, t) - \eta_1(x)\beta(x, t)w(x, t) \geq 0 & \text{on } \Gamma_1 \times (0, T], \\ \frac{\partial w}{\partial n}(x, t) = 0 & \text{on } \Gamma_2 \times (0, T], \\ w(x, 0) = 0 & \text{in } \Omega. \end{cases}$$

Applying Corollary B.8, we have $w \geq 0$ on $\overline{\Omega} \times [0, T]$. \square

Next, we turn to the existence of the solution. As a common process to deal with the nonlinear problem, we will take advantage of the theories for the linear problems and some fixed point theorems. Let $T > 0$, $R > 0$ and $X_T = C(\overline{\Omega} \times [0, T])$ be equipped with the maximum norm: $\|u\| \triangleq \max_{\overline{\Omega} \times [0, T]} |u|$ for any $u \in X_T$, then X_T is a Banach space and

$$X_{T,R} \triangleq \{v \in X_T : \|v\| \leq R\}$$

is also a Banach space. For any $v \in X_{T,R}$, it follows from Theorem B.6 and Corollary B.8 that there exists a unique solution $u \in \mathcal{A}_T$ to the following problem

$$\begin{cases} u_t(x, t) - \Delta u(x, t) = f(x, t) & \text{in } \Omega \times (0, T], \\ \frac{\partial u}{\partial n}(x, t) = \eta(x)F(v(x, t)) & \text{on } \Gamma_1 \times (0, T], \\ \frac{\partial u}{\partial n}(x, t) = 0 & \text{on } \Gamma_2 \times (0, T], \\ u(x, 0) = \psi(x) & \text{in } \Omega. \end{cases} \quad (\text{B.28})$$

Thus, it determines a mapping $\Psi_T : X_{T,R} \rightarrow \mathcal{A}_T$. Our strategy is to pick up a suitable R (depending on T) show that Ψ_T has a fixed point in $X_{T,R}$, which turns out to be the unique solution to (B.26).

In the proof of Theorem B.14, we will utilize the Schauder fixed point theorem, which requires to verify the following three things:

- (i) Ψ_T maps $X_{T,R}$ to $\mathcal{A}_T \cap X_{T,R}$ for some suitably chosen $R > 0$;
- (ii) $\Psi_T : X_{T,R} \rightarrow X_{T,R}$ is continuous;
- (iii) $\Psi_T(X_{T,R})$ is precompact in $X_{T,R}$.

Usually, the requirement (iii) is the most technical part and this time it requires the following Lemma B.13, which is a fact mentioned in the proof of ([7], Theorem 13, Sec. 5, Chap. 7).

Lemma B.13. *Given $T > 0$ and $\{\varphi_j\}_{j \geq 1} \subset L^\infty((\Gamma_1 \cup \Gamma_2) \times (0, T])$, we define*

$$w_j(x, t) = \int_0^t \int_{\partial\Omega} \Phi(x - y, t - \tau) \varphi_j(y, \tau) dS_y d\tau, \quad \forall (x, t) \in \overline{\Omega} \times [0, T]. \quad (\text{B.29})$$

If $\{\varphi_j\}_{j \geq 1}$ are uniformly bounded on $(\Gamma_1 \cup \Gamma_2) \times (0, T]$, then $\{w_j\}_{j \geq 1}$ are uniformly bounded and equicontinuous on $\overline{\Omega} \times [0, T]$.

Proof. By using the estimate

$$|\Phi(x - y, t - \tau)| \leq C (t - \tau)^{-3/4} |x - y|^{-(n-3/2)},$$

it is not hard to prove this Lemma, so we omit the proof here. \square

Now based on the arguments in ([7], Theorem 13, Sec. 5, Chap. 7) and ([13], Theorem 1.3), we conclude the following theorem on the existence of the solution.

Theorem B.14. *For the nonlinear problem (B.26) with f, η, F, ψ described there, we have the following two conclusions.*

- (1) *There exists $T_0 > 0$ such that for any $0 < T \leq T_0$, there exists a unique solution $u \in \mathcal{A}_T$ to (B.26) on $\overline{\Omega} \times [0, T]$.*
- (2) *If F is a bounded function on \mathbb{R} , then for any $T > 0$, there exists a unique solution $u \in \mathcal{A}_T$ to (B.26) on $\overline{\Omega} \times [0, T]$.*

Proof. Just as the heuristic idea before Lemma B.13, in order to prove the existence of a solution, we will use Schauder fixed point theorem to show Ψ_T has a fixed point in $X_{T,R}$ for some $R > 0$. Namely, we need to verify the following three requirements:

- (i) Ψ_T maps $X_{T,R}$ to $\mathcal{A}_T \cap X_{T,R}$ for some suitably chosen $R > 0$ (depending on T);
- (ii) $\Psi_T : X_{T,R} \rightarrow X_{T,R}$ is continuous;
- (iii) $\Psi_T(X_{T,R})$ is precompact in $X_{T,R}$.

In the following, we will prove (1) and (2) in Theorem B.14 together. Actually, the proofs of requirements (ii) and (iii) for (1) and (2) are identically the same, only the proofs of requirement (i) has slightly difference.

Firstly, given $T > 0$, let's recall how we construct $u \triangleq \Psi_T(v)$ for $v \in X_{T,R}$. We will use the same notations as those in the proof of Theorem B.6, but with $\beta = 0$ and $g(x, t) = \eta(x) F(v(x, t)) \mathbf{I}_{\Gamma_1}(x)$, where

$$\mathbf{I}_{\Gamma_1}(x) \triangleq \begin{cases} 1 & x \in \Gamma_1 \\ 0 & x \notin \Gamma_1 \end{cases}$$

is the indicator function. Thus u has the following expression: for any $(x, t) \in \overline{\Omega} \times [0, T]$,

$$\begin{aligned} u(x, t) \triangleq & \int_{\Omega_1} \Phi(x - y, t) \psi(y) dy + \int_0^t \int_{\Omega} \Phi(x - y, t - \tau) f(y, \tau) dy d\tau \\ & + \int_0^t \int_{\partial\Omega} \Phi(x - y, t - \tau) \varphi(y, \tau) dS_y d\tau. \end{aligned} \tag{B.30}$$

Here $\varphi \in \mathcal{B}_T$ satisfies for any $(x, t) \in (\Gamma_1 \cup \Gamma_2) \times (0, T]$,

$$\varphi(x, t) = \int_0^t \int_{\partial\Omega} K(x, t; y, \tau) \varphi(y, \tau) dS_y d\tau + H(x, t),$$

where

$$K(x, t; y, \tau) = -2 (D\Phi)(x - y, t - \tau) \cdot \vec{n}(x) \tag{B.31}$$

and

$$\begin{aligned}
H(x, t) = & -2 \int_{\Omega_1} (D\Phi)(x - y, t) \cdot \vec{n}(x) \psi(y) dy \\
& -2 \int_0^t \int_{\Omega} (D\Phi)(x - y, t - \tau) \cdot \vec{n}(x) f(y, \tau) dy d\tau \\
& + 2 \eta(x) F(v(x, t)) \mathbf{I}_{\Gamma_1}(x).
\end{aligned} \tag{B.32}$$

Because the function K in (B.31) also satisfies (B.4), we can apply Lemma B.4 and follow the same way as the derivations of (B.18), (B.19) to get

$$\varphi(x, t) = \int_0^t \int_{\partial\Omega} K^*(x, t; y, \tau) H(y, \tau) dS_y d\tau + H(x, t) \tag{B.33}$$

for some function K^* . Moreover, there exists a constant $C^* = C^*(n, \Omega, T)$ such that

$$|K^*(x, t; y, \tau)| \leq C^*(t - \tau)^{-3/4} |x - y|^{-(n-3/2)}. \tag{B.34}$$

Next, we will first assume requirement (i) and prove requirements (ii) and (iii), then we will confirm requirement (i) for the Cases (1) and (2) in Theorem B.14 respectively. Given $T > 0$, we assume there exists $R > 0$ such that $\Psi_T : X_{T,R} \rightarrow \mathcal{A}_T \cap X_{T,R}$. Define $M_F, M_{F'} : [0, \infty) \rightarrow \mathbb{R}$ by

$$M_F(r) = \sup_{|x| \leq r} |F(x)|$$

and

$$M_{F'}(r) = \sup_{|x| \leq r} |F'(x)|,$$

then both $M_F(r)$ and $M_{F'}(r)$ are finite for any $r \geq 0$, since $F \in C^1(\mathbb{R})$. In the following, for any $v \in X_{T,R}$, we write H , φ and u as defined in (B.32), (B.31) and (B.30) respectively. For any $v_j \in X_{T,R}$ ($j \geq 1$), we analogously write H_j , φ_j and u_j .

- Proof of Requirement (ii). Given $\{v_j\}_{j \geq 1} \subset X_{T,R}$ and $v_j \rightarrow v$ in $X_{T,R}$, we want to show $\Psi_T(v_j) \rightarrow \Psi_T(v)$ in $X_{T,R}$. Because v and all the v_j ($j \geq 1$) belong to $X_{T,R}$, then for any $(x, t) \in \overline{\Omega} \times [0, T]$, $|v(x, t)| \leq R$ and $|v_j(x, t)| \leq R$. Thus, by the mean value theorem and the fact $M_{F'}(R) < \infty$, it follows from (B.32) that $H_j \rightrightarrows H$ on $(\Gamma_1 \cup \Gamma_2) \times (0, T]$ (here “ \rightrightarrows ” means “converges uniformly to”). Then by (B.33) and (B.34), $\varphi_j \rightrightarrows \varphi$ on $(\Gamma_1 \cup \Gamma_2) \times (0, T]$. Finally, due to the expression (B.30), we have $u_j \rightrightarrows u$ on $\overline{\Omega} \times [0, T]$, which implies $u \in X_{T,R}$ and $\Psi_T(v_j) \rightarrow \Psi_T(v)$ in $X_{T,R}$.
- Proof of Requirement (iii). In this proof, we will use C to denote a constant which is independent of j , x and t , but may depend on $n, \Omega, \Omega_1, T, R, M_F(R), \sup |f|, \sup |\psi|, \sup |D\psi|$ and $\sup |\eta|$. C may be different from line to line. Given any sequence $\{v_j\}_{j \geq 1} \subset X_{T,R}$, we want to show $\{\Psi_T(v_j)\}_{j \geq 1}$ has a subsequence which converges to some function u in $X_{T,R}$. Since $v_j \in X_{T,R}$ for any $j \geq 1$, then for any $j \geq 1$ and for any $(x, t) \in \overline{\Omega} \times [0, T]$, $|v_j(x, t)| \leq R$. Recalling (B.13), we know

$$\left| \int_{\Omega_1} (D\Phi)(x - y, t) \cdot \vec{n}(x) \psi(y) dy \right|$$

is bounded by some constant C . As a result, based on (B.32), there exists another constant C such

that for any $j \geq 1$ and for any $(x, t) \in \overline{\Omega} \times [0, T]$,

$$|H_j(x, t)| \leq C.$$

Then due to (B.33) and (B.34), there exists some constant C such that for any $j \geq 1$ and for any $(x, t) \in \overline{\Omega} \times [0, T]$,

$$|\varphi_j(x, t)| \leq C.$$

Now using (B.30) and Lemma B.13, we find $\{u_j\}_{j \geq 1}$ is uniformly bounded and equicontinuous on $\overline{\Omega} \times [0, T]$. Hence, it follows from the Arzela-Ascoli theorem that $\{u_j\}_{j \geq 1}$ has a subsequence $\{u_{j_k}\}_{k \geq 1}$ which converges uniformly to some function u on $\overline{\Omega} \times [0, T]$. Since $u_{j_k} \in X_{T,R}$, it is readily to see that u is also in $X_{T,R}$. Thus, $\Psi_T(X_{T,R})$ is precompact in $X_{T,R}$.

Now we turn to verify Requirement (i).

- Proof of Requirement (i) for (1). We will find $0 < T_0 \leq 1$ such that for any $0 < T \leq T_0$, there exists $R > 0$ such that Ψ_T maps $X_{T,R}$ to $\mathcal{A}_T \cap X_{T,R}$. In this proof, C will denote a constant which is independent of x, t, R and T , but may depend on $n, \Omega, \Omega_1, \sup |f|, \sup |\psi|, \sup |D\psi|$ and $\sup |\eta|$. C may be different from line to line. For the first term of (B.32), we recall (B.13) again to get for any $(x, t) \in (\Gamma_1 \cup \Gamma_2) \times (0, T]$,

$$\begin{aligned} & \left| \int_{\Omega_1} (D\Phi)(x - y, t) \cdot \vec{n}(x) \psi(y) dy \right| \\ & \leq \left| \int_{\partial\Omega_1} \Phi(x - y, t) \psi(y) \vec{n}(y) dy \cdot \vec{n}(x) \right| + \left| \int_{\Omega_1} \Phi(x - y, t) D\psi(y) dy \cdot \vec{n}(x) \right| \\ & \leq C \int_{\partial\Omega_1} |x - y|^{-n} dy + C \leq C. \end{aligned} \tag{B.35}$$

For the second term of (B.32), we have for any $(x, t) \in (\Gamma_1 \cup \Gamma_2) \times (0, T]$,

$$\begin{aligned} & \left| \int_0^t \int_{\Omega} (D\Phi)(x - y, t - \tau) \cdot \vec{n}(x) f(y, \tau) dy d\tau \right| \\ & \leq C \sup |f| \int_0^t \int_{\Omega} (t - \tau)^{-3/4} |x - y|^{-(n-1/2)} dS_y d\tau \\ & \leq C t^{1/4} \leq C T^{1/4} \leq C T_0^{1/4} \leq C, \end{aligned} \tag{B.36}$$

where the last inequality uses the assumption $T_0 \leq 1$. Then it follows from (B.35), (B.36) and (B.32) that for any $(x, t) \in (\Gamma_1 \cup \Gamma_2) \times (0, T]$,

$$|H(x, t)| \leq C + C M_F(R). \tag{B.37}$$

Although the constant C^* in (B.34) depends on T , if one checks its proof, it is readily to see that C^* is an increasing function in T . As a result, when T is bounded by 1, C^* will also be bounded by some constant C , which only depends on n and Ω . Based on this observation and (B.33), we get for any

$$(x, t) \in (\Gamma_1 \cup \Gamma_2) \times (0, T],$$

$$\begin{aligned} |\varphi(x, t)| &\leq C^* [C + C M_F(R)] \int_0^t \int_{\partial\Omega} (t - \tau)^{-3/4} |x - y|^{-(n-3/2)} dS_y d\tau + C + C M_F(R) \\ &\leq [C + C M_F(R)] T^{1/4} + C + C M_F(R) \\ &\leq C + C M_F(R). \end{aligned} \tag{B.38}$$

Now by (B.30) and (B.38), we obtain for any $(x, t) \in \overline{\Omega} \times [0, T]$,

$$\begin{aligned} |u(x, t)| &\leq \sup |\psi| + t \sup |f| + C \sup |\varphi| \int_0^t \int_{\partial\Omega} (t - \tau)^{-3/4} |x - y|^{-(n-3/2)} dS_y d\tau \\ &\leq C + [C + C M_F(R)] C T^{1/4} \\ &\leq C + C M_F(R) T_0^{1/4} \\ &\triangleq C_1 + C_1 M_F(R) T_0^{1/4}. \end{aligned} \tag{B.39}$$

Hence, if we choose R and $T_0 \leq 1$ such that

$$R = 2C_1 \quad \text{and} \quad T_0^{1/4} M_F(2C_1) < 1, \tag{B.40}$$

then we have $\|u\| \leq R$ and therefore $u \triangleq \Psi_T(v) \in X_{T,R}$.

- Proof of Requirement (i) for (2). We will prove that for any $T > 0$, there exists $R > 0$ such that Ψ_T maps $X_{T,R}$ to $\mathcal{A}_T \cap X_{T,R}$. From the assumption, F is a bounded function in \mathbb{R} , so $\sup_{\mathbb{R}} |F| < \infty$. In the rest of this proof, we will use C to denote a constant just like that in the proof for (1) but additionally allowing C to depend on T and $\sup_{\mathbb{R}} |F|$. As the same derivations as (B.35), (B.36) and (B.37), we attain for any $(x, t) \in (\Gamma_1 \cup \Gamma_2) \times (0, T]$,

$$\begin{aligned} |H(x, t)| &\leq C + C M_F(R) \\ &\leq C + C \sup_{\mathbb{R}} |F| = C. \end{aligned} \tag{B.41}$$

Then based on (B.33) and (B.41), we get for any $(x, t) \in (\Gamma_1 \cup \Gamma_2) \times (0, T]$,

$$\begin{aligned} |\varphi(x, t)| &\leq C^* C \int_0^t \int_{\partial\Omega} (t - \tau)^{-3/4} |x - y|^{-(n-3/2)} dS_y d\tau + C \\ &\leq C T^{1/4} + C \\ &= C. \end{aligned}$$

As a result, by (B.30) and (B.38) again, we obtain for any $(x, t) \in \overline{\Omega} \times [0, T]$,

$$\begin{aligned} |u(x, t)| &\leq \sup |\psi| + t \sup |f| + C \sup |\varphi| \int_0^t \int_{\partial\Omega} (t - \tau)^{-3/4} |x - y|^{-(n-3/2)} dS_y d\tau \\ &\leq C + C T^{1/4} \\ &\triangleq C_2. \end{aligned}$$

Thus, as long as choosing $R > C_2$, we will have $\|u\| \leq R$ and consequently $u \triangleq \Psi_T(v) \in X_{T,R}$.

□

As we can see from Theorem B.10, the solution to (B.26) is proved to be global only under the assumption that the function F being bounded on \mathbb{R} . Thus, when F is unbounded, we need to consider the maximal solution and figure out when the solution can exist globally.

Definition B.15. *We call*

$$T^* \triangleq \sup\{T \geq 0 : \text{there exists a solution to (B.26) on } \overline{\Omega} \times [0, T]\}$$

to be the maximal existence time for (B.26). Moreover, a function $u^ \in C^{2,1}(\Omega \times (0, T^*)) \cap C(\overline{\Omega} \times [0, T^*))$ is called a maximal solution if it solves (B.26) on $\overline{\Omega} \times [0, T]$ for any $T \in (0, T^*)$.*

Remark B.16. *It follows from Theorem B.14 and Theorem B.10 that the maximal solution exists and is unique.*

In ([13], Corollary 1.1), it concludes that when T^* is finite, then it coincides with the blow-up time of the L^∞ norm of u^* . Actually, it is the same situation here. More precisely, we have the following Theorem B.17.

Theorem B.17. *Let T^* be the maximal existence time for (B.26) and u^* be the maximal solution. If $T^* < \infty$, then*

$$\sup_{\overline{\Omega} \times [0, T^*)} |u^*(x, t)| = \infty. \quad (\text{B.42})$$

Proof. We refer the readers to the arguments in ([13], Corollary 1.1). □

Thus, if we can prove the solution to be bounded all the time, then it exists globally. Moreover, in order to estimate T^* , one only needs to study the blow-up time for the L^∞ norm of the solution.

Remark B.18. *As a particular application of the theories established in this section, one can apply Theorem B.14, Theorem B.10, Theorem B.11 and Remark B.16 with $f = 0$, $\eta = 1$, $F(\lambda) = \lambda^q$ and $\psi = u_0$ to our targeted problem (1.1) to obtain Theorem 1.3.*

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